A NOTE ON THE COHOMOLOGY RINGS OF MATROID SCHUBERT VARIETIES

YAIRON CID-RUIZ

ABSTRACT. This note is prepared as a companion for a presentation in the event "Arbeitsgemeinschaft: Combinatorial Hodge Theory" at the Mathematisches Forschungsinstitut Oberwolfach (MFO), Germany. We aim to present a proof (as self-contained as possible) of the result of Huh and Wang [HW17] establishing that the cohomology ring of a matroid Schubert variety coincides with the graded Möbius algebra.

1. Introduction

Let M be a simple matroid on the set $[n] = \{1,\ldots,n\}$ which is *realizable* over $\mathbb{C}$. This means that there exists a linear subspace $L \subset \mathbb{C}^n$ such that the rank function of M is given by

$$\operatorname{rank}_M(S) \;=\; \dim\left(\Pi_S(L)\right) \quad \text{ for all } \quad S \subseteq [n],$$

where $\Pi_S : \mathbb{C}^n = \bigoplus_{j\in[n]} \mathbb{C}\cdot \mathbf{e}_j \to \mathbb{C}^S = \bigoplus_{j\in S} \mathbb{C}\cdot \mathbf{e}_j$ is the natural projection and $\mathbf{e}_j = (0,\ldots,1,\ldots,0)$ is the j-th elementary basis vector. As introduced by Ardila and Boocher [AB16], the *matroid Schubert variety* $Y_M = Y_{M,L}$ of M is the closure of L under the natural inclusions

$$L \hookrightarrow \mathbb{C}^n = \mathbb{C}^1 \times \cdots \times \mathbb{C}^1 \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = \left(\mathbb{P}^1\right)^n.$$

Let $\mathcal{L}_M^\bullet$ be the lattice of flats of M. For each flat $F \in \mathcal{L}_M^\bullet$, we introduce the symbol y_F. Consider the graded free $\mathbb{Z}$-module

$$B^\bullet(M) := \bigoplus_{i\geqslant 0} B^i(M) \quad \text{ where } \quad B^i(M) := \bigoplus_{F\in\mathcal{L}_M^i} \mathbb{Z}\cdot y_F.$$

We endow $B^\bullet(M)$ with the structure of a commutative graded algebra over $\mathbb{Z}$ by setting

$$y_{F_1}y_{F_2} \;=\; \begin{cases} y_{F_1\vee F_2} & \text{if } \operatorname{rank}_M(F_1)+\operatorname{rank}_M(F_2) = \operatorname{rank}_M(F_1\vee F_2)\\ 0 & \text{otherwise,}\end{cases}$$

and extending this by linearity. To simplify notation, we write $y_1,\ldots,y_n$ instead of $y_{\{1\}},\ldots,y_{\{n\}}$. Under the above product operation, $y_\emptyset = 1$ is the identity element and the equality $y_F = \prod_{i\in I_F} y_i$ holds for any basis I_F of the flat F. Therefore, we can see $B^\bullet(M)$ as a quotient of the polynomial ring $\mathbb{Z}[y_1,\ldots,y_n]$.

We are interested on the following remarkable result.

Theorem 1.1 (Huh-Wang [HW17, Theorem 14]; see Theorem 2.15)**.** *For a realizable matroid M, we have the isomorphism of $\mathbb{Z}$-algebras*

$$B^\bullet(M) \xrightarrow{\cong} H^{2\cdot\bullet}(Y_M,\mathbb{Z}), \quad y_i \mapsto h_i,$$

where h_i denotes the first Chern class of the line bundle $\mathcal{O}_{Y_M}(\mathbf{e}_i)$.

2. Proof of the theorem

As before, M is a matroid on $[n]$ which is realized by the linear subspace $L \subset \mathbb{C}^n$. Let $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1,\ldots,x_n]$ and $\mathbb{C}[\mathbf{x},\mathbf{z}] = \mathbb{C}[x_1,\ldots,x_n,z_1,\ldots,z_n]$ be the coordinate rings $\mathbb{C}^n$ and $\left(\mathbb{P}^1\right)^n$. To simplify notation, we often write $\mathbb{P} := \left(\mathbb{P}^1\right)^n$. Let $I(L) \subset \mathbb{C}[\mathbf{x}]$ be the vanishing ideal of the linear subspace $L \subset \mathbb{C}^n$. The vanishing ideal of Y_M can be computed by the multi-homogenization

$$I(Y_M) \;=\; \left(f^h \mid f \in I(L)\right) \;\subset\; \mathbb{C}[\mathbf{x},\mathbf{z}].$$

For any $f \in \mathbb{C}[\mathbf{x}]$, the multi-homogenization f^h is obtained by substituting $x_i \mapsto \frac{x_i}{z_i}$ and then clearing out denominators.

Remark 2.1. Let $X = V(f_1,\ldots,f_s) \subset \mathbb{C}^n$. In general, it may be difficult to compute the equations of the closure $Y = \overline{X}$ of X in $\left(\mathbb{P}^1\right)^n$. By saturating with respect to the variables $z_1,\ldots,z_n$, we obtain

$$Y \;=\; V\left(\left(f_1^h,\ldots,f_s^h\right) : \left(\prod_{i=1}^n z_i\right)^{\infty}\right).$$

Indeed, this can be deduced as follows. Let $Y' = V\left(f_1^h,\ldots,f_s^h\right) \subset \left(\mathbb{P}^1\right)^n$. Let $Z = V(z_1\cdots z_n) \subset \left(\mathbb{P}^1\right)^n$ and $j : U = \left(\mathbb{P}^1\right)^n \setminus Z \to \left(\mathbb{P}^1\right)^n$ be the natural immersion. We have that $\mathcal{H}_Z^0\left(\mathcal{O}_Y\right) = 0$ (the ideal $I(Y)$ of the closure $Y = \overline{X}$ is saturated with respect to $z_1,\ldots,z_n$) and that $\mathcal{O}_Y|_U \cong \mathcal{O}_{Y'}|_U$ (the dehomogenizations of both Y and Y' are both equal to X). Then we get the exact sequences

$$0 \to \mathcal{O}_Y \to j_*\left(\mathcal{O}_Y|_U\right) \to \mathcal{H}_Z^1\left(\mathcal{O}_Y\right) \to 0 \;\text{ and }\; 0 \to \mathcal{H}_Z^0\left(\mathcal{O}_{Y'}\right) \to \mathcal{O}_{Y'} \to j_*\left(\mathcal{O}_{Y'}|_U\right) \to \mathcal{H}_Z^1\left(\mathcal{O}_{Y'}\right) \to 0$$

involving local cohomology sheaves (see [Har67, Corollary 1.9]). By comparing both exact sequences, we obtain $\mathcal{O}_Y \cong \mathcal{O}_{Y'}/\mathcal{H}_Z^0\left(\mathcal{O}_{Y'}\right)$, as required.

Remark 2.2. For any circuit C of the matroid M, there is a linear form $\sum_{c\in C} a_c x_c$ in $I(L)$, which is unique up to multiplication by a nonzero scalar.

The following important result of Ardila and Boocher shows that the equations of Y_M are completely determined by the circuits of the matroid M.

Theorem 2.3 ([AB16, Theorem 1.3(a)])**.** *$Y_M \subset \left(\mathbb{P}^1\right)^n$ is defined by the multi-homogenization of the circuits of M. More precisely, we have*

$$Y_M \;=\; V\Big(\; \textstyle\sum_{c\in C} a_c x_c \prod_{d\in C\setminus\{c\}} z_d \;\Big|\; C \text{ is a circuit of } M \;\Big).$$

To compute the the cohomology ring of Y_M, we use Borel-Moore homology and a certain algebraic cell decomposition of Y_M into affine spaces. This homology theory is quite successful for noncompact topological spaces.

Let $Y \subset \left(\mathbb{P}^1\right)^n$ be an r-dimensional locally closed reduced subscheme[1]. A general fact is that Y can be a embedded as a closed subspace of some real space $\mathbb{R}^N$. Then the *Borel-Moore homology* of Y can be computed as

$$\overline{H}_i(Y) \;\cong\; H^{N-i}\left(\mathbb{R}^N, \mathbb{R}^N \setminus Y; \mathbb{Z}\right),$$

[1]We reserve the term *variety* for an integral and separated scheme of finite type over $\mathbb{C}$.

where the right hand side denotes relative singular cohomology with integer coefficients. For more details, see [Ful97, Appendix B] and [Ful98, §19.1]. For an irreducible k-dimensional subvariety $V \subset Y$, we obtain the *fundamental class*

$$[V] := \iota_*(\eta_V) \in \overline{H}_{2k}(Y),$$

where $\iota_* : \overline{H}_{2k}(V) \to \overline{H}_{2k}(Y)$ is pushforward map and η_V is the canonical generator of $\overline{H}_{2k}(V) = \mathbb{Z}$.

Remark 2.4. Borel-Moore homology coincides with singular homology for compact and locally contractible spaces. Therefore, if $Y \subset (\mathbb{P}^1)^n$ is a closed subvariety, then we obtain $\overline{H}_\bullet(Y) = H_\bullet(Y, \mathbb{Z})$.

Remark 2.5. By a standard abuse of notation, when Y is a smooth projective variety, we also denote by $[V]$ the fundamental class of V in $H^{2c}(Y, \mathbb{Z})$ where $c = \dim(Y) - \dim(V)$ is the codimension of V. That is, we take the image of $[V]$ under the (Poincaré duality) isomorphism $H_{2k}(Y, \mathbb{Z}) = \overline{H}_{2k}(Y) \xrightarrow{\cong} H^{2(r-k)}(Y, \mathbb{Z})$.

Remark 2.6. Recall that we have a natural short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(Y, \mathbb{Z}), \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_i(Y, \mathbb{Z}), \mathbb{Z}) \to 0$$

from the Universal Coefficient Theorem.

The next standard lemma will be the main tool in our approach.

Lemma 2.7 ([Ful97, Lemma 6, Appendix B])**.** *Let $Y = Y_m \supset Y_{m-1} \supset \cdots \supset Y_1 \supset Y_0 = \varnothing$ be a sequence of closed reduced subschemes. Assume that $Y_i \setminus Y_{i-1}$ is a disjoint union of varieties $U_{i,j}$ each isomorphic to an affine space $\mathbb{C}^{n(i,j)}$. Then the classes $\left[\overline{U}_{i,j}\right]$ of the closures of these varieties give an additive basis for the Borel-Moore homology groups $\overline{H}_\bullet(Y)$ over $\mathbb{Z}$.*

We use the convention $\mathbb{P}^1 = \mathbb{C} \cup \infty$ with $\infty = (1:0)$. Hence, for any $S \subseteq [n]$, the subvariety

$$U_S := V(z_j \mid j \notin S) \setminus V(z_j \mid j \in S) \subset (\mathbb{P}^1)^n$$

can be identified with $U_S \cong \left(\prod_{j \in S} \mathbb{C}\right) \times \left(\prod_{j \notin S} \infty\right) \cong \mathbb{C}^{|S|}$. More explicitly, in terms of coordinates, we have

$$U_S \cong \mathrm{Spec}\left(\mathbb{C}\left[w_j \mid j \in S\right]\right) \times \left(\prod_{j \notin S} \mathrm{Proj}(\mathbb{C}[x_j])\right) \cong \mathrm{Spec}\left(\mathbb{C}\left[w_j \mid j \in S\right]\right) \tag{1}$$

where $w_i = x_i / z_i$. The closure of U_S in $(\mathbb{P}^1)^n$ is equal to

$$(\mathbb{P}^1)^S := \left(\prod_{j \in S} \mathbb{P}^1\right) \times \left(\prod_{j \notin S} \infty\right) \subset (\mathbb{P}^1)^n.$$

Let $Y_{n+1} := (\mathbb{P}^1)^n$ and $Y_0 := \varnothing$, and for $1 \leqslant i \leqslant n$, let

$$\begin{aligned} Y_i &:= V\left(\bigcap_{S \subseteq [n] \text{ and } |S| = n+1-i} \left(z_j \mid j \in S\right)\right) \subset (\mathbb{P}^1)^n \\ &= \bigcup_{S \subseteq [n] \text{ and } |S| = i-1} (\mathbb{P}^1)^S. \end{aligned}$$

This is a sequence of closed reduced subschemes $\left(\mathbb{P}^1\right)^n = Y_{n+1} \supset Y_n \supset \cdots \supset Y_1 \supset Y_0 = \varnothing$ and a simple computation shows that

$$Y_{i+1} \setminus Y_i \;=\; \bigsqcup_{S \subseteq [n] \text{ and } |S| = i} U_S. \tag{2}$$

Remark 2.8. From Lemma 2.7, we can deduce the well-known result that

$$H^{2\cdot\bullet}\left(\left(\mathbb{P}^1\right)^n, \mathbb{Z}\right) \;\cong\; \frac{\mathbb{Z}[h_1,\ldots,h_n]}{\left(h_1^2,\ldots,h_n^2\right)} \tag{3}$$

where

$$h_i \;=\; \big[\mathbb{P}^1 \times \cdots \times \underbrace{\infty}_{i-\text{th}} \times \cdots \mathbb{P}^1\big].$$

Proof. By Lemma 2.9, (2) and the Universal Coefficient Theorem, the classes $\left[\left(\mathbb{P}^1\right)^S\right]$ give a $\mathbb{Z}$-basis of $H^{2\cdot\bullet}(\mathbb{P},\mathbb{Z})$. It remains to determine the cup product on $H^{2\cdot\bullet}(\mathbb{P},\mathbb{Z})$. Let $Z_i := V(z_i) \subset \mathbb{P}$ and $Z_i' := V(x_i) \subset \mathbb{P}$. Let $S \subseteq [n]$ and write $[n] \setminus S = \{i_1,\ldots,i_c\}$. Since $\left(\mathbb{P}^1\right)^S = Z_{i_1} \cap \cdots \cap Z_{i_c}$ can be obtained as a sequence of transversal intersections, it follows that

$$\left[\left(\mathbb{P}^1\right)^S\right] \;=\; [Z_{i_1}] \smile \cdots \smile [Z_{i_c}] \;=\; h_{i_1} \cdots h_{i_c};$$

see [Ful97, page 213, eq. (9)]. Since Z_i and Z_i' are rationally equivalent, [Ful98, Proposition 19.1.1] implies that $h_i = [Z_i] = \left[Z_i'\right] \in H^2(\mathbb{P},\mathbb{Z})$. Consequently, we obtain the vanishing

$$h_i \cdot h_i \;=\; [Z_i] \smile \left[Z_i'\right] = 0$$

because $Z_i \cap Z_i' = \varnothing$. This completes the proof. ∎

Let $Y_M^i = Y_M \cap Y_i$ and consider the sequence

$$Y_M \;=\; Y_M^{n+1} \;\supset\; Y_M^n \;\supset\; \cdots \;\supset\; Y_M^1 \;\supset\; Y_M^0 = \emptyset.$$

As a consequence of Theorem 2.3 we obtain the following.

Lemma 2.9. *We have the equality*

$$Y_M^{i+1} \setminus Y_M^i \;\cong\; \bigsqcup_{F \in \mathcal{L}_M^\bullet \text{ and } |F| = i} \mathbb{C}^{\mathrm{rank}_M(F)}.$$

Moreover, for any $S \subseteq [n]$, we have

$$U_S \cap Y_M \;=\; \begin{cases} \mathbb{C}^{\mathrm{rank}_M(S)} & \text{if } S \text{ is a flat of } M \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. By intersecting (2) with Y_M, we obtain

$$Y_M^{i+1} \setminus Y_M^i \;=\; \bigsqcup_{S \subseteq [n] \text{ and } |S| = i} U_S \cap Y_M.$$

Let C be a circuit of M and $F_C = \sum_{c \in C} a_c x_c \prod_{d \in C \setminus \{c\}} z_d$ be the corresponding multi-homogeneous polynomial vanishing on Y_M; by Theorem 2.3, these polynomials determine Y_M. For any subset $S \subseteq [n]$, F_C yields a regular function on U_S (see (1)). We have the following three possibilities:

(i) If $C \subseteq S$, then F_C yields the linear form $\sum_{c \in C} a_c w_c$ on U_S.
(ii) If $|C \setminus S| = 1$, then $V(F_C) \cap U_S = \varnothing$.
(iii) If $|C \setminus S| \geqslant 1$, then $V(F_C) \supset U_S$.

Finally, the result of the lemma follows from Remark 2.10 below. □

Remark 2.10. A subset $F \subseteq [n]$ is a flat of the matroid M if and only if $|C \setminus F| \neq 1$ for any circuit C of the matroid M.

Proof. ($\Rightarrow$) Suppose F is a flat. Assume by contradiction that there exists a circuit C with $|C \setminus F| = 1$. Write $C \setminus F = \{c\}$. Then $C \setminus \{c\} \subseteq F$. This implies $c \in \mathrm{cl}(F) = F$, a contradiction.

($\Leftarrow$) Let $F \subseteq [n]$ such that $|C \setminus F| \neq 1$ for every circuit C. We must show that F is a flat. Take any $e \in [n] \setminus F$. Assume by contradiction that $e \in \mathrm{cl}(F)$. Then there is a circuit C with $e \in C \subseteq F \cup \{e\}$. Therefore $C \setminus F = \{e\}$, a contradiction. □

By combining the previous results, we already get a basis for the cohomology ring of the matroid Schubert variety Y_M.

Corollary 2.11. *In odd degrees: for all $i \geqslant 0$, we have $H_{2i+1}(Y_M, \mathbb{Z}) = 0$ and $H^{2i+1}(Y_M, \mathbb{Z}) = 0$. In even degrees: for all $i \geqslant 0$, we have*

$$H_{2i}(Y_M, \mathbb{Z}) \cong \bigoplus_{F \in \mathcal{L}_M^i} \mathbb{Z} \cdot \left[\overline{U_F \cap Y_M}\right] \quad \text{and} \quad H^{2i}(Y_M, \mathbb{Z}) \cong \bigoplus_{F \in \mathcal{L}_M^i} \mathbb{Z} \cdot \xi_F,$$

where ξ_F is the dual of the basis element $\left[\overline{U_F \cap Y_M}\right] \in \overline{H}_{2i}(Y_M) = H_{2i}(Y_M, \mathbb{Z})$.

Proof. Due Lemma 2.7, Remark 2.4 and Lemma 2.9, it follows that $H_\bullet(Y_M, \mathbb{Z})$ is generated freely as a $\mathbb{Z}$-module by the elements $\left[\overline{U_F \cap Y_M}\right]$ for F a flat of M. On the other hand, since $H_\bullet(Y_M, \mathbb{Z})$ is $\mathbb{Z}$-free, the Universal Coefficient Theorem yields a natural isomorphism $H^i(Y_M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(Y_M, \mathbb{Z}), \mathbb{Z})$. □

For the rest of the note, let $r := \mathrm{rank}(M)$ be the rank of the matroid M. We also need the following result by Ardila and Boocher.

Theorem 2.12 ([AB16, Theorem 1.3(c)])**.** *The fundamental class of Y_M in $\mathbb{P} = \left(\mathbb{P}^1\right)^n$ is given by*

$$[Y_M] = \sum_{B \text{ is a basis of } M} \left[\left(\mathbb{P}^1\right)^B\right] \in H_{2r}(\mathbb{P}, \mathbb{Z}).$$

Following Brion [Bri03], we say that Y_M is a *multiplicity-free* variety. In fact, by utilizing [Bri03, Theorem 0.1], we obtain that Y_M is normal and arithmetically Cohen-Macaulay and, most importantly, that it admits a flat degeneration to the reduced union

$$\bigcup_{B \text{ is a basis of } M} \left(\mathbb{P}^1\right)^B \subset \left(\mathbb{P}^1\right)^n$$

of products of $\mathbb{P}^1$'s.

Let F be a flat of M. Notice that $U_F \cap Y_M$ is isomorphic to the linear space $\Pi_F(L)$ and that $\left[\overline{U_F \cap Y_M}\right]$ is the corresponding matroid Schubert variety in $\left(\mathbb{P}^1\right)^F$. Hence the class of $\left[\overline{U_F \cap Y_M}\right]$ in $\mathbb{P} = \left(\mathbb{P}^1\right)^n$ is given by

$$\left[\overline{U_F \cap Y_M}\right] = \sum_{B \text{ is a basis of } F} \left[\left(\mathbb{P}^1\right)^B\right] \in H_{2\,\mathrm{rank}_M(F)}(\mathbb{P}, \mathbb{Z}). \tag{4}$$

We need the following technical lemma.

Lemma 2.13. *Let $A^\bullet$ be a graded algebra $\mathbb{Z}$-algebra that is a finite free $\mathbb{Z}$-module. Assume the following conditions:*

(a) $\operatorname{rank}\left(A^i\right) = |\mathcal{L}_M^i|$ *for all* i.
(b) *Let* $\mathbb{Z}[y_1,\ldots,y_n]$ *be a standard graded polynomial ring (i.e.,* $\deg(y_i)=1$*). Let* $C^\bullet = \mathbb{Z}[y_1,\ldots,y_n]/\mathcal{J}$*, where* $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ *is the sum of ideals*

$$\mathcal{J}_1 \;=\; \left(y_1^{a_1}\cdots y_n^{a_n} \mid \sum_{s\in S} a_s > \operatorname{rank}_M(S) \text{ for some } S\subseteq[n]\right)$$

and

$$\mathcal{J}_2 \;=\; \left(y_{i_1}\cdots y_{i_k} - y_{j_1}\cdots y_{j_k} \mid \{i_1,\ldots,i_k\} \textit{ and } \{j_1,\ldots,j_k\} \textit{ are bases of the same flat of } M\right).$$

(c) *There is a graded surjection* $\pi : C^\bullet \twoheadrightarrow A^\bullet$.

Then we actually have an isomorphism $\pi : C^\bullet \xrightarrow{\cong} A^\bullet$.

Proof. Consider the exact sequence $0\to K\to C^\bullet\to A^\bullet\to 0$. Let $\Bbbk$ be a field. Since $A^\bullet$ is $\mathbb{Z}$-flat, we have $\operatorname{Tor}_1^{\mathbb{Z}}(A^\bullet,\Bbbk)=0$, and so we get a short exact sequence

$$0 \;\to\; K\otimes_{\mathbb{Z}}\Bbbk \;\to\; C^\bullet\otimes_{\mathbb{Z}}\Bbbk \;\to\; A^\bullet\otimes_{\mathbb{Z}}\Bbbk \;\to\; 0.$$

One can check that both graded $\Bbbk$-algebras $C^\bullet\otimes_{\mathbb{Z}}\Bbbk$ and $A^\bullet\otimes_{\mathbb{Z}}\Bbbk$ have the same Hilbert function (also, see Remark 2.14). Therefore $K\otimes_{\mathbb{Z}}\Bbbk = 0$ for any field $\Bbbk$. This shows that $K=0$, as required. □

Remark 2.14. Let $\Bbbk$ be a field and consider the polynomial ring $\Bbbk[y_1,\ldots,y_n]$. The set of polynomials

$$\begin{aligned}\mathcal{G} \;=\; &\left\{y_1^2,\ldots,y_n^2\right\} \cup \left\{y_{i_1}\cdots y_{i_k} \mid \{i_1,\ldots,i_k\} \text{ is a dependent set of } M\right\}\\ &\cup \left\{y_{i_1}\cdots y_{i_k} - y_{j_1}\cdots y_{j_k} \mid \{i_1,\ldots,i_k\} \text{ and } \{j_1,\ldots,j_k\} \text{ are bases of the same flat of } M\right\}\end{aligned}$$

gives a universal Gröbner basis of the ideal $\mathcal{J}\otimes_{\mathbb{Z}}\Bbbk \subset \Bbbk[y_1,\ldots,y_n]$ determined by the ideal $\mathcal{J}\subset\mathbb{Z}[y_1,\ldots,y_n]$ in Lemma 2.13. For details, see [LMMP25, Proposition 3.1].

Finally, we ready for the proof of the main result of this note.

Theorem 2.15. *For a realizable matroid* M*, we have the isomorphism of* $\mathbb{Z}$*-algebras*

$$B^\bullet(M) \;\xrightarrow{\cong}\; H^{2\cdot\bullet}(Y_M,\mathbb{Z}), \quad y_i \mapsto h_i,$$

where h_i *denotes the first Chern class of the line bundle* $\mathcal{O}_{Y_M}(\mathbf{e}_i)$.

Proof. Let $C^\bullet = \mathbb{Z}[y_1,\ldots,y_n]/\mathcal{J}$ be the graded algebra of Lemma 2.13. Notice that we have a graded surjective map $C^\bullet \twoheadrightarrow B^\bullet(M)$. Due to Lemma 2.13, we obtain the isomorphism $C^\bullet \xrightarrow{\cong} B^\bullet(M)$.

Let $\iota : Y_M \hookrightarrow \mathbb{P} = \left(\mathbb{P}^1\right)^n$ be the closed immersion. From Corollary 2.11 and (4), we obtain that the pushforward map $\iota_* : H_\bullet(Y_M,\mathbb{Z}) \hookrightarrow H_\bullet(\mathbb{P},\mathbb{Z})$ is injective. Hence the Universal Coefficient Theorem implies that the pullback map

$$\iota^* \;:\; H^\bullet(\mathbb{P},\mathbb{Z}) \;\twoheadrightarrow\; H^\bullet(Y_M,\mathbb{Z})$$

is surjective. The cohomology ring of $\mathbb{P} = (\mathbb{P}^1)^n$ is isomorphic to $\mathbb{Z}[h_1, \ldots, h_n]/(h_1^2, \ldots, h_n^2)$ (see (3)). Let α be a class in $H^i(\mathbb{P}, \mathbb{Z})$. By the Universal Coefficient Theorem, we have the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P},\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(H_0(\mathbb{P},\mathbb{Z}),\mathbb{Z}) \\ \downarrow \smile \alpha & & \downarrow (\frown \alpha)^* \\ H^i(\mathbb{P},\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(H_i(\mathbb{P},\mathbb{Z}),\mathbb{Z}) \\ \downarrow \iota^* & & \downarrow (\iota_*)^* \\ H^i(Y_M,\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(H_i(Y_M,\mathbb{Z}),\mathbb{Z}). \end{array}$$

Therefore, $\iota^*(\alpha) = 0$ if and only if $\alpha \frown \iota_*(\beta) = 0$ for all $\beta \in H_i(Y_M, \mathbb{Z})$.

For any $h_{i_1} \cdots h_{i_k} \in H^{2k}(\mathbb{P}, \mathbb{Z})$ and any flat F of M of rank k, the equation (4) yields

$$h_{i_1} \cdots h_{i_k} \frown \left[\overline{U_F \cap Y_M}\right] = \begin{cases} 1 & \text{if } \{i_1, \ldots, i_k\} \text{ is a basis of } F \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 2.11, the classes $\left[\overline{U_F \cap Y_M}\right]$ with $\mathrm{rank}_M(F) = k$ give a $\mathbb{Z}$-basis of $H_{2k}(Y_M, \mathbb{Z})$. Therefore the sets of elements

$$\left\{ h_1^{a_1} \cdots h_n^{a_n} \mid \sum_{s \in S} a_s > \mathrm{rank}_M(S) \text{ for some } S \subseteq [n] \right\}$$

and

$$\left\{ h_{i_1} \cdots h_{i_k} - h_{j_1} \cdots h_{j_k} \mid \{i_1, \ldots, i_k\} \text{ and } \{j_1, \ldots, j_k\} \text{ are bases of the same flat of } M \right\}$$

lie in the kernel of ι^*. Consequently, we obtain a graded surjective map

$$C^\bullet \twoheadrightarrow H^{2\cdot\bullet}(Y_M, \mathbb{Z}), \quad y_i \mapsto h_i.$$

Finally, Corollary 2.11 and Lemma 2.13 yield the isomorphism $C^\bullet \xrightarrow{\cong} H^{2\cdot\bullet}(Y_M, \mathbb{Z})$. □

Remark 2.16. For the case of polymatroids, Crowley, Simpson and Wang [CSW24] gave a suitable generalization of Theorem 2.15 by utilizing the notions of *polymatroid Schubert varieties* and *combinatorial flats* (which they introduced).

REFERENCES

[AB16] Federico Ardila and Adam Boocher, *The closure of a linear space in a product of lines*, J. Algebraic Combin. **43** (2016), no. 1, 199–235. 1, 2, 5

[Bri03] Michel Brion, *Multiplicity-free subvarieties of flag varieties*, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 13–23. 5

[CSW24] Colin Crowley, Connor Simpson, and Botong Wang, *Polymatroid schubert varieties*, arXiv preprint arXiv:2410.10552 (2024). 7

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry. 3, 4

[Ful98] ———, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. 3, 4

[Har67] Robin Hartshorne, *Local cohomology*, Lecture Notes in Mathematics, vol. No. 41, Springer-Verlag, Berlin-New York, 1967, A seminar given by A. Grothendieck, Harvard University, Fall, 1961. 2

[HW17] June Huh and Botong Wang, *Enumeration of points, lines, planes, etc.*, Acta Math. **218** (2017), no. 2, 297–317. 1

[LMMP25] Adam LaClair, Matthew Mastroeni, Jason McCullough, and Irena Peeva, *Koszul graded Möbius algebras and strongly chordal graphs*, Selecta Math. (N.S.) **31** (2025), no. 2, Paper No. 29, 30. 6

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695, USA
Email address: ycidrui@ncsu.edu