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Blow-up algebras in Algebra, Geometry and Combinatorics

by

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To mom and dad,
of course.

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Contents

Introduction

Abstract

The primary topic of this thesis lies at the crossroads of Commutative Algebra and its interactions with Algebraic Geometry and Combinatorics. It is mainly focused around the following themes:

$\boxed{\text{I}}$ Defining equations of blow-up algebras.

$\boxed{\text{II}}$ Study of rational maps via blow-up algebras.

$\boxed{\text{III}}$ Asymptotic properties of the powers of edge ideals of graphs.

We are primarily interested in questions that arise in geometrical or combinatorial contexts and try to understand how their possible answers manifest in various algebraic structures or invariants. There is a particular algebraic object, the Rees algebra (or blow-up algebra), that appears in many constructions of Commutative Algebra, Algebraic Geometry, Geometric Modeling, Computer Aided Geometric Design and Combinatorics. The workhorse and main topic of this doctoral dissertation has been the study of this algebra under various situations.

The Rees algebra was introduced in the field of Commutative Algebra in the famous paper [121]. Since then, it has become a central and fundamental object with numerous applications. The study of this algebra has been so fruitful that it is difficult to single out particular results or papers, instead we refer the reader to the books [144] and [146] to wit the "landscape of blow-up algebras".

From a geometrical point of view, the Rees algebra corresponds with the bi-homogeneous coordinate ring of two fundamental objects: the blow-up of a projective variety along a subvariety and the graph of a rational map between projective varieties (see [66, §II.7]). Therefore, the importance of finding the defining equations of the Rees algebra is probably beyond argument. This is a problem of tall order that has occupied commutative algebraists and algebraic geometers, and despite an extensive effort (see [14, 20, 34–36, 38, 39, 78, 100, 103–105, 108, 113, 143]), it remains open even in the case of polynomial rings in two variables. In [29], Chapter 2 of this dissertation,

we use the theory of D-modules to describe the defining ideal of the Rees algebra in the case of a parametrization of a plane curve.

The study of rational and birational maps is classical in the literature from both an algebraic and geometric point of view, and it goes back to the work of Cremona [41], at least. A relatively new idea, probably first used in [80], is to look at the syzygies of the base ideal of a rational map to determine birationality. This algebraic method for studying rational maps has now become an active research topic (see [16, 46, 51, 67, 68, 102, 119, 124, 129]). In a joint work with Busé and D'Andrea [22], Chapter 3 of this dissertation, we introduce a new algebra that we call the saturated special fiber ring, which turns out to be an important tool to analyze the degree of a rational map. Later, in [30], Chapter 4 of this dissertation, we compute the multiplicity of this new algebra in the case of perfect ideals of height two, which, in particular, provides an effective method to determine the degree of a rational map having those ideals as base ideal.

Often a good tactic to approach a challenging problem is to go all the way up to a generic case and then find sufficient conditions for the specialization to keep some of the main features of the former. The procedure depends on taking a dramatic number of variables to allow modifying the given data into a generic shape, and usually receives the name of specialization. This method is seemingly due to Kronecker and was quite successful in the hands of Hurwitz ([86]) in establishing a new elegant theory of elimination and resultants. More recent instances where specialization is used are, e.g., [84], [85], [141], [132]. In a joint work with Simis [33], Chapter 5 of this dissertation, we consider the behavior of the degree of a rational map under specialization of the coefficients of the defining linear system.

The Rees algebra of the edge ideal of a graph is a well studied object (see [54, 56, 136, 147–150]), that relates combinatorial properties of a graph with algebraic invariants of the powers of its edge ideal. For the Rees algebra of the edge ideal of a bipartite graph, in [31], Chapter 6 of this dissertation, we compute the universal Gröbner basis of its defining equations and its total Castelnuovo-Mumford regularity as a bigraded algebra.

It is a celebrated result that the regularity of the powers of a homogeneous ideal is asymptotically a linear function (see [42, 99]). Considerable efforts have been put forth to understand the form of this asymptotic linear function in the case of edge ideals (see [3, 4, 8–10, 62, 90]). In a joint work with Jafari, Picone and Nemati [32], Chapter 7 of this dissertation, for bicyclic graphs, i.e. graphs containing exactly two cycles, we characterize the regularity of its edge ideal in terms of the induced matching number and determine the previous asymptotic linear function in special cases.

The basic outline of this thesis is as follows. In Chapter 1, we recall some preliminary results and definitions to be used along this work. Then, the thesis is divided in three different parts. The first part corresponds with the theme "$\boxed{\text{I}}$ *Defining equations of blow-up algebras*" and consists of Chapter 2. The second part corresponds with the theme "$\boxed{\text{II}}$ *Study of rational maps via blow-up algebras*" and consists of Chapter 3, Chapter 4 and Chapter 5. The third part corresponds with the theme "$\boxed{\text{III}}$ *Asymptotic properties of the powers of edge ideals of graphs*" and consists of Chapter

6 and Chapter 7. The common thread and main tool in the three parts of this thesis is the use of blow-up algebras.

In the subsequent sections of the introduction, we describe the motivations, organization, and main contributions and results of this dissertation.

$\boxed{\mathrm{I}}$ Defining equations of blow-up algebras

Let $\mathbb{F}$ be a field of characteristic zero, R be the polynomial ring $R = \mathbb{F}[x_1, x_2]$, and $I = (f_1, f_2, f_3) \subset R$ be a height two ideal minimally generated by three homogeneous polynomials of the same degree d. The Rees algebra of I is defined as $\mathcal{R}(I) = R[It] = \bigoplus_{i=0}^{\infty} I^i t^i$. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, T_2, T_3]$ via the map

$$S = R[T_1, T_2, T_3] \xrightarrow{\psi} \mathcal{R}(I), \quad \psi(T_i) = f_i t.$$

Of particular interest are the defining equations of the Rees algebra $\mathcal{R}(I)$, that is, the kernel $\mathcal{J} = \mathrm{Ker}(\psi)$ of this map ψ. A large number of works have been done to determine the equations of the Rees algebra, and the problem has been studied by algebraic geometers and commutative algebraists under various conditions (see e.g. [144] and the references therein). In recent years, a lot of attention has been given to find the minimal generators of the equations of the Rees algebra for an ideal in a polynomial ring (see e.g. [20, 34–36, 38, 39, 79, 100, 103–105, 108]), partly inspired by new connections with Geometric Modeling. Despite this extensive effort, even in the "simple" case studied in Chapter 2, the problem of finding the minimal generators of $\mathcal{J}$ remains open.

By the Hilbert-Burch Theorem (see e.g. [47, Theorem 20.15]) we know that the presentation of I is of the form

$$0 \to R(-d-\mu) \oplus R(-2d+\mu) \xrightarrow{\varphi} R(-d)^3 \xrightarrow{[f_1, f_2, f_3]} I \to 0,$$

and I is generated by the 2×2-minors of φ; we may assume that $0 < \mu \leqslant d - \mu$. The symmetric algebra of I can easily be described as a quotient of S by using the presentation of I. The defining equations of the symmetric algebra are given by

$$[g_1, g_2] = [T_1, T_2, T_3] \cdot \varphi,$$

and so we have $\mathrm{Sym}(I) \cong S/(g_1, g_2)$. There is an important relation between $\mathrm{Sym}(I)$ and $\mathcal{R}(I)$ in the form of the following canonical exact sequence

$$0 \to \mathcal{K} \to \mathrm{Sym}(I) \xrightarrow{\alpha} \mathcal{R}(I) \to 0.$$

Here we have $\mathcal{K} = \mathcal{J}/(g_1, g_2)$, which allows us to take $\mathcal{K}$ as the object of study.

The main novelty of Chapter 2 is the use of D-modules to find different descriptions of $\mathcal{K}$. In Chapter 2, we prove that $\mathcal{K}$ can be described as the solution set of a system of differential equations, that the whole bigraded structure of $\mathcal{K}$ is characterized by the integral roots of certain b-functions, and that certain de Rham cohomology groups can give partial information about $\mathcal{K}$.

The polynomial ring S is bigraded with $\mathrm{bideg}(T_i) = (1, 0)$ and $\mathrm{bideg}(x_i) = (0, 1)$, and $\mathcal{K}$, $\mathrm{Sym}(I)$ and $\mathcal{R}(I)$ have natural structures of bigraded S-modules. Let $\mathcal{T}$ be the polynomial ring $\mathcal{T} = A_2(\mathbb{F})[T_1, T_2, T_3]$ over the Weyl algebra $A_2(\mathbb{F})$ and consider the differential operators $L_1 = \mathcal{F}(g_1)$ and $L_2 = \mathcal{F}(g_2)$ by applying the Fourier transform to g_1 and g_2 (see Definition 2.14).

The first main result of Chapter 2 gives that $\mathcal{K}$ can be described by solving a system of differential equations.

Theorem A (Theorem 2.18). *Let $I \subset R = \mathbb{F}[x_1, x_2]$ be a height two ideal minimally generated by three homogeneous polynomials of the same degree d, and g_1 and g_2 be the defining equations of the symmetric algebra of I. Let $L_1 = \mathcal{F}(g_1)$ and $L_2 = \mathcal{F}(g_2)$ be the Fourier transform of g_1 and g_2, respectively. Then, we have the following isomorphism of bigraded S-modules*

$$\mathcal{K} \cong \mathrm{Sol}\Big(L_1, L_2; S\Big)_{\mathcal{F}}(-2, -d+2),$$

where $\mathrm{Sol}(L_1, L_2; S) = \{h \in S \mid L_1 \bullet h = 0 \text{ and } L_2 \bullet h = 0\}$ and the subscript-$\mathcal{F}$ denotes the twisting by the Fourier transform (Section 2.2).

Since g_1 and g_2 generate all the linear part of $\mathcal{J}$ (the syzygies of I) and $\mathcal{K} = \mathcal{J}/(g_1, g_2)$, then we have $\mathcal{K}_{p,*} = 0$ for all $p < 2$. As an application of Theorem A we give a complete characterization of the graded structure of each R-module $\mathcal{K}_{p,*}$ ($p \geqslant 2$) in terms of the integral roots of certain b-functions (Definition 2.26).

Theorem B (Theorem 2.31). *Let $I \subset R = \mathbb{F}[x_1, x_2]$ be as in Theorem A. Then, for each integer $p \geqslant 2$ there exists a nonzero b-function $b_p(s)$, and we have a relation between the graded structure of $\mathcal{K}_{p,*}$ and the integral roots of $b_p(s)$ given in the following equivalence*

$$\mathcal{K}_{p,q} \neq 0 \qquad \Longleftrightarrow \qquad b_p(-d+2+q) = 0.$$

Even more, we have that these are the only possible roots of $b_p(s)$, that is

$$b_p(s) = \prod_{\{q \in \mathbb{Z} \mid \mathcal{K}_{p,q} \neq 0\}} (s + d - 2 - q).$$

Theorem B is interesting for us in the sense that it gives a tool for deducing information about $\mathcal{K}$, but on the other hand, from a D-module point of view it is worthy to note that it describes the b-function of a family of holonomic D-modules like those in Setup 2.21.

Let U be the polynomial ring $U = \mathbb{F}[T_1, T_2, T_2]$.

In the last main result of Chapter 2, we change the role of L_1 and L_2, more specifically, instead of having them as operators we place them in a quotient. We make this change by means of a duality proven in Theorem 2.32, which allows us to establish an isomorphism of graded U-modules between $\mathcal{K}$ and a certain de Rham cohomology group. In particular, this isomorphism could give an alternative way to compute or estimate the dimension $\dim_{\mathbb{F}}(\mathcal{K}_{p,*})$ of each $\mathcal{K}_{p,*}$ regarded as finite dimensional $\mathbb{F}$-vector space (see Corollary 2.6).

Theorem C (Theorem 2.37)**.** *Let* $I \subset R = \mathbb{F}[x_1, x_2]$, L_1 *and* L_2 *be as in Theorem A, and let* Q *be the left* $\mathcal{T}$*-module* $Q = \mathcal{T}/\mathcal{T}(L_1, L_2)$*. Then, we have the following isomorphism of graded* U*-modules*

$$\mathcal{K} \cong H^0_{dR}(Q) = \{w \in Q \mid \partial_1 \bullet w = 0 \text{ and } \partial_2 \bullet w = 0\}.$$

In particular, for any integer p *we have an isomorphism of* $\mathbb{F}$*-vector spaces*

$$\mathcal{K}_{p,*} \cong H^0_{dR}(Q_p) = \{w \in Q_p \mid \partial_1 \bullet w = 0 \text{ and } \partial_2 \bullet w = 0\}.$$

The basic outline of Chapter 2 is as follows. In Section 2.1, we prove Theorem 2.5. In Section 2.2, we make a translation of our problem into the theory of D-modules and we prove Theorem A. In Section 2.3, we prove Theorem B. In Section 2.4, we prove Theorem C. In Section 2.5, we present a script in *Macaulay2* [60] that can compute each b-function $b_p(s)$ from Theorem B, and using it we effectively recover the bigraded structure of $\mathcal{K}$ for a couple of examples.

$\boxed{\text{II}}$ Study of rational maps via blow-up algebras

Questions and results concerning the degree and birationality of rational maps are classical in the literature from both an algebraic and geometric point of view. These problems have been extensively studied since the work of Cremona [41] in 1863 and are still very active research topics (see e.g. [2, 45, 46, 64, 80, 129] and the references therein). However, there seems to be very few results and no general theory available for multi-graded rational maps, i.e. maps that are defined by a collection of multi-homogeneous polynomials over a subvariety of a product of projective spaces. At the same time, there is an increasing interest in those maps, for both theoretic and applied purposes (see e.g. [15, 16, 77, 127, 128]).

In Chapter 3, we give formulas and effective sharp bounds for the degree of multi-graded rational maps and provide some effective criteria for birationality in terms of their algebraic and geometric properties. Sometimes we also improve known results in the single-graded case. Our approach is based on the study of blow-up algebras, including syzygies, of the ideal generated by the defining polynomials of a rational map, which is called the base ideal of the map. This idea goes back to [80] and since then a large amount of papers has blossomed in this direction (see e.g. [16, 46, 51, 67, 68, 102, 119, 124, 129]).

One of the main contributions in Chapter 3 is the introduction of a new algebra that we call the *saturated special fiber ring* (see Definition 3.3). The fact that saturation plays a key role in this kind of problems has already been observed in previous works in the single-graded setting (see [67], [92, Theorem 3.1], [118, Proposition 1.2]). Based on that, we define this new algebra by taking certain multi-graded parts of the saturation of all the powers of the base ideal. It can be seen as an extension of the more classical special fiber ring. We show that this new algebra turns out to be a fundamental ingredient for computing the degree of a rational map (see Theorem 3.4), emphasizing in this way that saturation has actually a prominent role to control the degree and birationality of a rational map. In particular, it allows us to prove a new general numerical formula (see Corollary 3.12) from which the degree of a rational map can be extracted. Another interesting feature of the saturated special fiber ring is its relation with the j-multiplicity of an ideal (see Lemma 3.10).

Next, we refine the above results for rational maps whose base locus is zero-dimensional. We begin by providing a purely algebraic proof (see Theorem 3.16) of the classical degree formula in intersection theory (see e.g. [55, Section 4.4]) adapted to our setting. Then, we investigate the properties that can be extracted if the syzygies of the base ideal of a rational map are used. This idea amounts to use the symmetric algebra to approximate the Rees algebra and it allows us to obtain sharp upper bounds (see Theorem 3.21) for the degree of multi-graded rational maps. Some applications to more specific cases of multi-projective rational maps (see Proposition 3.24) and projective rational maps (see Theorem 3.34) are also discussed, with the goal of providing efficient birationality criteria in the low-degree cases (see e.g. Proposition 3.27, Theorem 3.28 and Proposition 3.35).

The problem of detecting whether a rational map is birational has attracted a lot of attention in the past thirty years. A typical example is the class of Cremona maps (i.e. birational) in the projective plane that have been studied extensively (see e.g. [2]). Obviously, the computation of the degree of a rational map yields a way of testing its birationality. Nevertheless, this approach is not very efficient and various techniques have been developed in order to improve it and to obtain finer properties of birational maps. Among these specific techniques, the Jacobian dual criterion introduced and fully developed in [46, 124, 129] has its own interest. In Section 3.3, we extend the theory of the Jacobian dual criterion to the multi-graded setting (see Theorem 3.39) and derive some consequences where the syzygies of the base ideal are used instead of the higher-order equations of the Rees algebra (see Theorem 3.44).

In Chapter 3, we also study the particular class of plane rational maps whose base ideal is saturated and has a syzygy of degree one. In this setting, we provide a very effective birationality criterion (see Theorem 3.59) and a complete description of the equations of the associated Rees algebra (see Theorem 3.57).

Now, we fix a basic notation in order to describe some results of Chapter 3 in more detail.

Let $\mathbb{k}$ be a field, $X_1 \subset \mathbb{P}_{\mathbb{k}}^{r_1}, X_2 \subset \mathbb{P}_{\mathbb{k}}^{r_2}, \ldots, X_m \subset \mathbb{P}_{\mathbb{k}}^{r_m}$ and $Y \subset \mathbb{P}_{\mathbb{k}}^{s}$ be integral projective varieties over $\mathbb{k}$. For $i = 1, \ldots, m$, the homogeneous coordinate ring of X_i is denoted by $A_i = \mathbb{k}[\mathbf{x_i}]/\mathfrak{a}_i = \mathbb{k}[x_{i,0}, x_{i,1}, \ldots, x_{i,r_i}]/\mathfrak{a}_i$, and $S = \mathbb{k}[y_0, y_1, \ldots, y_s]/\mathfrak{b}$ stands for the homogeneous coordinate ring of Y. Set $R = A_1 \otimes_{\mathbb{k}} A_2 \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} A_m \cong \mathbb{k}[\mathbf{x}]/(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$. Let

$$\mathcal{F} : X = X_1 \times_{\mathbb{k}} X_2 \times_{\mathbb{k}} \cdots \times_{\mathbb{k}} X_m \dashrightarrow Y \subset \mathbb{P}_{\mathbb{k}}^{s}$$

be a dominant rational map defined by $s + 1$ multi-homogeneous elements $\mathbf{f} = \{f_0, f_1, \ldots, f_s\} \subset R$ of the same multi-degree $\mathbf{d} = (d_1, d_2, \ldots, d_m)$. We also assume that X is an integral variety.

The degree of the dominant rational map $\mathcal{F} : X = X_1 \times_{\Bbbk} X_2 \times_{\Bbbk} \cdots \times_{\Bbbk} X_m \dashrightarrow Y$ is defined as $\deg(\mathcal{F}) = [K(X) : K(Y)]$, where $K(X)$ and $K(Y)$ represent the fields of rational functions of X and Y, respectively.

The ring $R = A_1 \otimes_{\Bbbk} A_2 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_m$ has a natural multi-grading given by

$$R = \bigoplus_{(j_1,\ldots,j_m)\in\mathbb{Z}^m} (A_1)_{j_1} \otimes_{\Bbbk} (A_2)_{j_2} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} (A_m)_{j_m}.$$

Let $\mathfrak{N}$ be the multi-homogeneous irrelevant ideal of R, that is

$$\mathfrak{N} = \bigoplus_{j_1>0,\ldots,j_m>0} R_{j_1,\ldots,j_m}.$$

For an arbitrary ideal $J \subset R$, let J^{sat} be the ideal $(J : \mathfrak{N}^\infty)$. Let T be the multi-Veronese subring which is given by the standard graded $\Bbbk$-algebra

$$T = \Bbbk[R_{\mathbf{d}}] = \bigoplus_{n=0}^{\infty} R_{n\cdot\mathbf{d}}.$$

The homogeneous coordinate ring S is often called the special fiber ring in the literature, and using the canonical graded homomorphism associated to $\mathcal{F}$ we obtain the identification

$$S \cong \Bbbk[f_0, f_1, \ldots, f_s] = \Bbbk[I_{\mathbf{d}}] = \bigoplus_{n=0}^{\infty} [I^n]_{n\cdot\mathbf{d}}$$

In Chapter 3, we define the following saturated version of S.

Definition D (Definition 3.3)**.** *The saturated special fiber ring of* I *is the graded* S*-algebra*

$$\widetilde{\mathfrak{F}_R(I)} = \bigoplus_{n=0}^{\infty} \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}}.$$

Interestingly, the algebra $\widetilde{\mathfrak{F}_R(I)}$ turns out to be finitely generated as an S-module.

The main result of Chapter 3 is given in the following theorem. It shows, for instance, that a comparison between the multiplicities of S and $\widetilde{\mathfrak{F}_R(I)}$ yields the degree of $\mathcal{F}$, and that under some conditions $\mathcal{F}$ is birational if and only if $S = \widetilde{\mathfrak{F}_R(I)}$.

Theorem E (Theorem 3.4)**.** *Let* $\mathcal{F} : X = X_1 \times_{\Bbbk} X_2 \times_{\Bbbk} \cdots \times_{\Bbbk} X_m \dashrightarrow Y$ *be a dominant rational map. If* $\dim(Y) = \dim(X)$*, then we have the following commutative diagram*

$$\begin{tikzcd}
X \arrow[d, "\cong"'] \arrow[drr, dashed, "\mathcal{F}"] & & \\
\mathrm{Proj}(T) \arrow[rr, dashed, "\mathcal{F}'"] \arrow[dr, dashed, "\mathcal{G}"'] & & Y \\
& \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right) \arrow[ur, "\mathcal{H}"'] &
\end{tikzcd}$$

where the maps $\mathcal{F}' : \mathrm{Proj}(T) \dashrightarrow Y$, $\mathcal{G} : \mathrm{Proj}(T) \dashrightarrow \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right)$ *and* $\mathcal{H} : \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right) \to Y$ *are induced from the inclusions* $S \hookrightarrow T$, $\widetilde{\mathfrak{F}_R(I)} \hookrightarrow T$ *and* $S \hookrightarrow \widetilde{\mathfrak{F}_R(I)}$, *respectively.*

Also, the statements below are satisfied:

(i) $\mathcal{H} : \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right) \to Y$ *is a finite morphism with* $\deg(\mathcal{F}) = \deg(\mathcal{H})$.

(ii) $\mathcal{G}$ *is a birational map.*

(iii) $e\left(\widetilde{\mathfrak{F}_R(I)}\right) = \deg(\mathcal{F}) \cdot e(S)$, *where* $e(-)$ *stands for multiplicity.*

(iv) Under the additional condition of S *being integrally closed, then* $\mathcal{F}$ *is birational if and only if* $\widetilde{\mathfrak{F}_R(I)} = S$.

We remark that all the hypotheses and consequences in the previous theorem do not depend on the characteristic of the field $\mathbb{k}$. By applying Theorem E, we give formulas and effective sharp bounds for the degree of multi-graded rational maps and provide some criteria for birationality in terms of their algebraic and geometric properties.

An example of such a criterion for birationality is the following characterization of birational maps from a multi-projective space onto a projective space, which in particular applies to Cremona transformations.

Corollary F (Proposition 3.24)**.** *Let* $\mathcal{F} : \mathbb{P}_{\mathbb{k}}^{r_1} \times_{\mathbb{k}} \mathbb{P}_{\mathbb{k}}^{r_2} \times_{\mathbb{k}} \cdots \times_{\mathbb{k}} \mathbb{P}_{\mathbb{k}}^{r_m} \dashrightarrow \mathbb{P}_{\mathbb{k}}^{\delta}$ *be a dominant rational map with* $r_1 + r_2 + \ldots + r_m = \delta$. *Then, the map* $\mathcal{F}$ *is birational if and only if for all* $n \geqslant 1$ *we have*

$$[I^n]_{n\cdot\mathbf{d}} = \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}}.$$

An example of such an upper bound for the degree of a rational map is the following.

Proposition G (Proposition 3.35)**.** *Let* $\mathcal{F} : \mathbb{P}_{\mathbb{k}}^2 \dashrightarrow \mathbb{P}_{\mathbb{k}}^2$ *be a dominant rational map with a dimension* 1 *base ideal* I *minimally generated by three polynomials of degree* d. *Then, the following statements hold:*

(i) $\deg(\mathcal{F}) \leqslant \frac{(d-1)(d-2)}{2} + \dim_{\mathbb{k}}\left([I^{\mathrm{sat}}/I]_d\right) + 1.$

(ii) *If* I *is of linear type and* $d \leqslant 3$, *then* $\deg(\mathcal{F}) = \frac{(d-1)(d-2)}{2} + \dim_{\mathbb{k}}\left([I^{\mathrm{sat}}/I]_d\right) + 1$.

Now, we briefly describe the relation of the saturated special fiber ring with the j-multiplicity of an ideal. The j-multiplicity of an ideal was introduced in [1] and serves as a generalization of the Hilbert-Samuel multiplicity for non $\mathfrak{m}$-primary ideals. It has applications in intersection theory (see [53]), and the problem of finding formulas for it has been addressed in several papers (see e.g. [91, 92, 114, 120]). Let A be a standard graded $\mathbb{k}$-algebra of dimension $\delta + 1$ which is an integral domain. Let $\mathfrak{m}$ be its maximal irrelevant ideal $\mathfrak{m} = A_+$ and let J^{sat} be $(J : \mathfrak{m}^\infty)$ for any ideal $J \subset A$. For a non necessarily $\mathfrak{m}$-primary ideal $J \subset A$ its j-multiplicity is given by

$$j(J) = \delta! \lim_{n\to\infty} \frac{\dim_{\mathbb{k}}\left(H^0_{\mathfrak{m}}\left(J^n/J^{n+1}\right)\right)}{n^\delta}.$$

Lemma H (Lemma 3.10)**.** *Let* $J \subset A$ *be a homogeneous ideal equally generated in degree* d. *Suppose* J *has maximal analytic spread* $\ell(J) = \delta + 1$. *Then, we have the equality*

$$j(J) = d \cdot e\left(\widetilde{\mathfrak{F}_A(J)}\right),$$

where $\widetilde{\mathfrak{F}_A(J)} = \bigoplus_{n=0}^{\infty}\left[(J^n)^{\mathrm{sat}}\right]_{nd}$ *is the saturated special fiber ring of* J.

Another sought interest of Chapter 3 is to develop a multi-graded version of the Jacobian dual criterion of [46]. This remarkable criterion gives necessary and sufficient conditions to test the birationality of a rational map, and, also, it should be noted that does not depend on the characteristic of the field $\mathbb{k}$. Additionally, when the map is birational, we can get the inverse map.

The following theorem contains a multi-graded version of the Jacobian dual criterion.

Theorem I (Theorem 3.39)**.** *Let* $\mathcal{F} : X_1 \times_{\mathbb{k}} X_2 \times_{\mathbb{k}} \cdots \times_{\mathbb{k}} X_m \dashrightarrow Y$ *be a dominant rational map. Let* ψ *and* ψ_i *for* $1 \leqslant i \leqslant m$ *be the Jacobian dual matrices of Notation 3.37. Then, the following three conditions are equivalent:*

(i) $\mathcal{F}$ *is birational.*

(ii) $\mathrm{rank}_S(\psi_i \otimes_{\mathbb{k}[\mathbf{y}]} S) = r_i$ *for each* $i = 1, \ldots, m$.

(iii) $\mathrm{rank}_S(\psi \otimes_{\mathbb{k}[\mathbf{y}]} S) = r_1 + r_2 + \cdots + r_m$.

In addition, if $\mathcal{F}$ *is birational then its inverse is of the form* $\mathcal{G} : Y \dashrightarrow X_1 \times_{\mathbb{k}} X_2 \times_{\mathbb{k}} \cdots \times_{\mathbb{k}} X_m$, *where each map* $Y \dashrightarrow X_i$ *is given by the signed ordered maximal minors of an* $r_i \times (r_i + 1)$ *submatrix of* ψ_i *of rank* r_i.

The basic outline of Chapter 3 is as follows. In Section 3.1, we introduce the saturated special fiber ring and we prove Theorem E. In Section 3.2, we study rational maps with a finite base locus. In Section 3.3, we extend the Jacobian dual criterion to the multi-graded setting and we prove Theorem I. In Section 3.4, we study a particular class of plane rational maps.

The case of perfect ideals of height two

In Chapter 4, we compute the multiplicity of the saturated special fiber ring for a general family of perfect ideals of height two. Interestingly, this formula is equal to an elementary symmetric polynomial in terms of the degrees of the syzygies of the ideal. As two simple corollaries, for this class of ideals, we obtain a closed formula for the j-multiplicity and an effective method for determining the degree and birationality of rational maps defined by homogeneous generators of these ideals.

Let $\mathbb{k}$ be a field, R be the polynomial ring $R = \mathbb{k}[x_0, x_1, \ldots x_r]$, and $\mathfrak{m}$ be the maximal irrelevant ideal $\mathfrak{m} = (x_0, x_1, \ldots, x_r)$. Let $I \subset R$ be a perfect ideal of height two which is minimally generated by $s+1$ forms $\{f_0, f_1, \ldots, f_s\}$ of the same degree d.

To determine the multiplicity of $\widetilde{\mathfrak{F}_R(I)}$, we need to study the first local cohomology module of the Rees algebra of I, and for this we assume the condition G_{r+1}. The condition G_{r+1} means that $\mu(I_\mathfrak{p}) \leqslant \dim(R_\mathfrak{p})$ for every non-maximal ideal $\mathfrak{p} \in V(I) \subset \mathrm{Spec}(R)$, where $\mu(I_\mathfrak{p})$ denotes the minimal number of generators of $I_\mathfrak{p}$. To study the Rees algebra one usually tries to reduce the problem in terms of the symmetric algebra, the assumption of G_{r+1} is important in making possible this reduction. After reducing the problem in terms of the symmetric algebra, we consider certain Koszul complex that provides an approximate resolution (see e.g. [101], [24]) of the symmetric algebra, and which permits us to compute the Hilbert series of $\widetilde{\mathfrak{F}_R(I)}$. By pursuing this general approach, we obtain the following theorem which is the main result of Chapter 4.

Theorem J (Theorem 4.8). *Let $I \subset R = \mathbb{k}[x_0, x_1, \ldots, x_r]$ be a homogeneous ideal minimally generated by $s+1$ forms $\{f_0, f_1, \ldots, f_s\}$ of the same degree d, where $s \geqslant r$. Suppose the following two conditions:*

(i) I *is perfect of height two with Hilbert-Burch resolution of the form*

$$0 \to \bigoplus_{i=1}^{s} R(-d-\mu_i) \to R(-d)^{s+1} \to I \to 0.$$

(ii) I *satisfies the condition G_{r+1}, that is $\mu(I_\mathfrak{p}) \leqslant \dim(R_\mathfrak{p})$ for all $\mathfrak{p} \in V(I) \subset \mathrm{Spec}(R)$ such that $\mathrm{ht}(\mathfrak{p}) < r+1$.*

Then, the multiplicity of the saturated special fiber ring $\widetilde{\mathfrak{F}_R(I)}$ of I is given by

$$e\left(\widetilde{\mathfrak{F}_R(I)}\right) = e_r(\mu_1, \mu_2, \ldots, \mu_s),$$

where $e_r(\mu_1, \mu_2, \ldots, \mu_s)$ represents the r-th elementary symmetric polynomial

$$e_r(\mu_1, \mu_2, \ldots, \mu_s) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant s} \mu_{j_1}\mu_{j_2}\cdots\mu_{j_r}.$$

The following result gives a formula for the j-multiplicity of a whole family of ideals.

Corollary K (Corollary 4.10)**.** *Assume all the hypotheses and notations of Theorem J. Then, the* j*-multiplicity of* I *is given by*

$$j(I) = d \cdot e_r(\mu_1, \mu_2, \ldots, \mu_s).$$

In the second application of Theorem J, we study the degree of a rational map $\mathcal{F} : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$ defined by the tuple of forms $\{f_0, f_1, \ldots, f_s\}$. We show that the product of the degree of $\mathcal{F}$ and the degree of the image of $\mathcal{F}$ is equal to $e_r(\mu_1, \ldots, \mu_s)$. From this we can determine the degree of a rational map by just computing the degree of the image, and conversely, the degree of the map gives us the degree of the image. In particular, we obtain that the map is birational if and only if the degree of the image is the maximum possible.

Corollary L (Corollary 4.11)**.** *Assume all the hypotheses and notations of Theorem J. Let* $\mathcal{F}$ *be the rational map* $\mathcal{F} : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$ *given by*

$$(x_0 : \cdots : x_r) \mapsto \big(f_0(x_0, \ldots, x_r) : \cdots : f_s(x_0, \ldots, x_r)\big),$$

and $Y \subset \mathbb{P}^s_{\Bbbk}$ *be the closure of the image of* $\mathcal{F}$*. Then, the following two statements hold:*

(i) $\deg(\mathcal{F}) \cdot \deg_{\mathbb{P}^s_{\Bbbk}}(Y) = e_r(\mu_1, \mu_2, \ldots, \mu_s)$.

(ii) $\mathcal{F}$ *is birational onto its image if and only if* $\deg_{\mathbb{P}^s_{\Bbbk}}(Y) = e_r(\mu_1, \mu_2, \ldots, \mu_s)$.

The basic outline of Chapter 4 is as follows. In Section 4.1, we prove Theorem J. In Section 4.2, we study rational maps whose base ideals satisfy the conditions of Theorem J.

Specialization of rational maps

The overall goal of Chapter 5 is to obtain bounds for the degree of a rational map in terms of the main features of its base ideal. In order that this objective stays within a reasonable limitation, we focus on rational maps whose source and target are projective varieties.

Now, to become more precise we should rather talk about projective schemes as source and target of the envisaged rational maps. The commonly sought interest is the case of projective schemes over a field (typically, but not necessarily, algebraically closed).

One tactic that has often worked is to go all the way up to a generic case and then find sufficient conditions for the specialization to keep some of the main features of the former. The procedure depends on taking a dramatic number of variables to allow modifying the given data into a generic shape. The method is seemingly due to Kronecker and was quite successful in the hands of Hurwitz ([86]) in establishing a new elegant theory of elimination and resultants. Of a more recent crop, we have, e.g., [84], [85], [141], [132].

In a related way, we have the notion of when an ideal specializes modulo a regular sequence: given an ideal $I \subset R$ in a ring, we say that I specializes with respect to a sequence of elements

$\{a_1, \ldots, a_n\} \subset R$ if the latter is a regular sequence both on R and on R/I. A tall question in this regard is to find conditions under which the defining ideal of some well-known rings – such as the Rees ring or the associated graded ring of an ideal (see, e.g., [49], [97]) – specialize with respect to a given sequence of elements. Often, at best we can only describe some obstructions to this sort of procedure, normally in terms of the kernel of the specialization map.

The core of Chapter 5 can be said to lie in between the two ideas of specialization as applied to the situation of rational maps between projective schemes and their related ideal-theoretic objects.

It so happens that at the level of the generic situation the coefficients live in a polynomial ring A over a field, not anymore on a field. This entails the need to consider rational maps defined by linear systems over the ring A, that is, rational maps with source $\mathbb{P}^r_A$. As it turns out, it is not exceedingly more complicated to consider rational maps with source an integral closed subscheme of $\mathbb{P}^r_A$.

Much to our surprise a complete such theory, with all the required details that include the ideal-theoretic transcription, is not easily available. For this reason, the first part of Chapter 5 deals with such details with an eye for the ideal-theoretic behavior concealed in or related to the geometric facts. A tall order in these considerations will be a so-called *relative fiber cone* that mimics the notion of a fiber cone (or special fiber ring; see Remark 5.4) in the classical environment over a field – this terminology is slightly misleading as the notion is introduced in algebraic language, associated to the concept of a Rees algebra rather than to the geometric version (blowup); however, we will draw on both the algebraic and the geometric versions.

Another concept dealt with is the *relative saturated fiber cone*, an object perhaps better understood in terms of global sections of a suitable sheaf of rings. In Chapter 5, the saturated special fiber ring (Definition D) is extended to the case when the coefficients belong to a Noetherian integral domain of finite Krull dimension. It contains the relative fiber cone as a subalgebra and plays a role in rational maps (see Theorem 5.25).

With the introduction of these considerations, we will be equipped to tackle the problem of specialization of rational maps, which is the main objective of Chapter 5. The neat application so far is to the multiplicity of the saturated fiber cone and to the degree of a rational map defined by the maximal minors of a homogeneous $(r+1) \times r$ matrix, when in both situations we assume that the coefficient ring A is a polynomial ring over a field of characteristic zero.

Another important result is that under a suitable general specialization of the coefficients, we show that the degree of the rational map never decreases, that the degree of the corresponding image never increases, but that the product of the two previous degrees remains constant.

Next there is a more detailed summary of the main results of Chapter 5.

Here we assume that the ground ring is a polynomial ring $A = \mathbb{k}[z_1, \ldots, z_m]$ over a field $\mathbb{k}$. In this setting we specialize the variables z_i to elements of $\mathbb{k}$. Thus, we consider a maximal ideal of the form $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ where $\alpha_i \in \mathbb{k}$. Since clearly $\mathbb{k} \cong A/\mathfrak{n}$, the A-module structure of $\mathbb{k}$ is given via the homomorphism $A \twoheadrightarrow A/\mathfrak{n} \cong \mathbb{k}$. We take a standard graded polynomial ring $R = A[x_0, \ldots, x_r]$ ($[R]_0 = A$) and a tuple of forms $\{g_0, \ldots, g_s\} \subset R$ of the same positive degree.

Let $\{\overline{g_0}, \ldots, \overline{g_s}\} \subset R/\mathfrak{n}R$ denote the corresponding tuple of forms in $R/\mathfrak{n}R \cong \Bbbk[x_0, \ldots, x_r]$ where $\overline{g_i}$ is the image of g_i under the canonical homomorphism $R \twoheadrightarrow R/\mathfrak{n}R$.

Consider the rational maps

$$\mathcal{G} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A \quad \text{ and } \quad \mathfrak{g} : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$$

determined by the tuples of forms $\{g_0, \ldots, g_s\}$ and $\{\overline{g_0}, \ldots, \overline{g_s}\}$, respectively.

The main target is finding conditions under which the degree $\deg(\mathfrak{g})$ of $\mathfrak{g}$ can be bounded above or below by the degree $\deg(\mathbb{G})$ of $\mathbb{G}$. Set $\mathcal{J} = (g_0, \ldots, g_s) \subset R$ and $I = (\overline{g_0}, \ldots, \overline{g_s}) \subset R/\mathfrak{n}R$. Let $\mathbb{E}(\mathcal{J})$ be the exceptional divisor of the blow-up of $\mathbb{P}^r_A$ along $\mathcal{J}$. A bit surprisingly, having a grip on the dimension of the scheme $\mathbb{E}(\mathcal{J}) \times_A \Bbbk$ is the main condition to determine whether $\deg(\mathfrak{g}) \leqslant \deg(\mathcal{G})$ or $\deg(\mathfrak{g}) \geqslant \deg(\mathcal{G})$. The main result in this direction is the following theorem.

Theorem M (Theorem 5.44)**.** *Suppose that both $\mathcal{G}$ and $\mathfrak{g}$ are generically finite.*

(i) Assume that the following conditions hold:

- *(a)* $\operatorname{Proj}(A[\mathbf{g}])$ *is a normal scheme.*
- *(b)* $\dim\left(\mathbb{E}(\mathcal{J}) \times_A \Bbbk\right) \leqslant r$.
- *(c)* $\Bbbk$ *is a field of characteristic zero.*

Then

$$\deg(\mathfrak{g}) \leqslant \deg(\mathcal{G}).$$

(ii) If $\dim\left(\mathbb{E}(\mathcal{J}) \times_A \Bbbk\right) \leqslant r - 1$, *then*

$$\deg(\mathfrak{g}) \geqslant \deg(\mathcal{G}).$$

(iii) Assuming that $\Bbbk$ is algebraically closed, there exists an open dense subset $\mathcal{W} \subset \Bbbk^m$ such that, if $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ with $(\alpha_1, \ldots, \alpha_m) \in \mathcal{W}$, then we have

$$\deg(\mathfrak{g}) \geqslant \deg(\mathcal{G}).$$

(iv) Consider the following condition:
$(\mathfrak{K})$ $k \geqslant 0$ *is a given integer such that* $\ell(\mathcal{J}_{\mathfrak{P}}) \leqslant \operatorname{ht}(\mathfrak{P}/\mathfrak{n}R) + k$ *for every prime ideal* $\mathfrak{P} \in \operatorname{Spec}(R)$ *containing* $(\mathcal{J}, \mathfrak{n})$.

Then:

- $(\mathfrak{K}1)$ *If* $(\mathfrak{K})$ *holds with* $k \leqslant 1$, *then condition (b) of part (i) is satisfied.*
- $(\mathfrak{K}2)$ *If* $(\mathfrak{K})$ *holds with* $k = 0$, *then the assumption of (ii) is satisfied.*

An additional interest in this chapter is the specialization of the saturated fiber cone of $\mathcal{J}$. By letting the specialization be suitably general, we prove that the multiplicity of the saturated special fiber ring of I is equal to the one of the saturated special fiber ring of $\mathcal{J} \otimes_A \mathrm{Quot}(A)$, where $\mathrm{Quot}(A)$ denotes the field of fractions of A. As a consequence, when the coefficients of the forms $\{\overline{g_0}, \ldots, \overline{g_s}\}$ are general, we obtain a formula for the product of the degree of $\mathbb{g}$ and the degree of the image of $\mathbb{g}$.

Let $\mathbb{K} = \mathrm{Quot}(A)$ denote the field of fractions of A and let $\mathbb{T} = \mathbb{K}[x_0, \ldots, x_r]$ denote the standard polynomial ring over $\mathbb{K}$ obtained from $R = A[x_0, \ldots, x_r]$ by base change (i.e., considering the A-coefficients of a polynomial as $\mathbb{K}$-coefficients). In addition, let $\mathbb{G}$ denote the rational map $\mathbb{G} : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow \mathbb{P}^s_{\mathbb{K}}$ defined by the tuple of forms $\{G_0, \ldots, G_s\}$, where G_i is the image of g_i along the canonical inclusion $R \hookrightarrow \mathbb{T}$. Finally, let $Y \subset \mathbb{P}^s_{\mathbb{k}}$ and $\mathbb{Y} \subset \mathbb{P}^s_{\mathbb{K}}$ be the closures of the images of $\mathbb{g}$ and $\mathbb{G}$, respectively. The following theorem contains the second main result of Chapter 5.

Theorem N (Theorem 5.47)**.** *Suppose that both $\mathcal{G}$ and $\mathbb{g}$ are generically finite. Assuming that $\mathbb{k}$ is algebraically closed, there exists an open dense subset $\mathcal{V} \subset \mathbb{k}^m$ such that, if $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ with $(\alpha_1, \ldots, \alpha_m) \in \mathcal{V}$, then we have*

$$\deg(\mathbb{g}) \cdot \deg_{\mathbb{P}^s_{\mathbb{k}}}(Y) = \deg(\mathbb{G}) \cdot \deg_{\mathbb{P}^s_{\mathbb{K}}}(\mathbb{Y}).$$

As a consequence of Theorem M(iii) and Theorem N we obtain the following corollary.

Corollary O (Corollary 5.48)**.** *Suppose that both $\mathcal{G}$ and $\mathbb{g}$ are generically finite. Assuming that $\mathbb{k}$ is algebraically closed, there exists an open dense subset $\mathcal{Q} \subset \mathbb{k}^m$ such that, if $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ with $(\alpha_1, \ldots, \alpha_m) \in \mathcal{Q}$, then we have*

$$\deg_{\mathbb{P}^s_{\mathbb{k}}}(Y) \leqslant \deg_{\mathbb{P}^s_{\mathbb{K}}}(\mathbb{Y}).$$

The basic outline of Chapter 5 is as follows. In Section 5.1, we fix some terminologies and notations. In Section 5.2, we develop in an algebraic fashion the main points of the theory of rational maps with source and target projective varieties defined over an arbitrary Noetherian integral domain of finite Krull dimension. In Section 5.3, we gather a few algebraic tools to be used in the specialization of rational maps. In Section 5.4, we prove Theorem M, Theorem N and Corollary O. In Section 5.5, we consider the problem of specialization of rational maps in the particular case of perfect base ideals of height two.

$\boxed{\mathrm{III}}$ Asymptotic properties of the powers of edge ideals of graphs

Let I be a homogeneous ideal in a polynomial ring $R = \mathbb{k}[x_1, \ldots, x_r]$ over a field $\mathbb{k}$. The Castelnuovo-Mumford regularity of I, denoted by $\mathrm{reg}(I)$, has been an interesting and active research topic for the past decades. There exists a vast literature on the study of $\mathrm{reg}(I)$ (see e.g. [25]). A celebrated result on the behavior of the regularity of powers of ideals was proved independently in [42] and [99]. In both papers, by making a detailed study of the Rees algebra of I, it is shown that for all $q \geqslant q_0$, the regularity of the powers of I is asymptotically a linear function $\mathrm{reg}(I^q) = dq + b$, where q_0 is the so-called stabilizing index, and b is the so-called constant. The value of d in the above formula is well understood (see [139, Theorem 3.2]). For example, d is equal to the degree of the generators of I when I is equigenerated in degree d (see loc. cit.). However, there is no general or precise method to determine q_0 and b.

In recent years, many researchers have tried to compute q_0 and b for special families of ideals. One of the simplest cases, yet interesting and wide open, is when I is the edge ideal of a finite simple graph. Let $G = (V(G), E(G))$ be a finite simple graph on the vertex set $V(G) = \{x_1, \dots, x_r\}$. The edge ideal $I = I(G)$, associated to G, is the ideal of R generated by the set of monomials $x_i x_j$ such that x_i is adjacent to x_j.

The problem of determining the stabilizing index and the constant have been settled for special families of graphs. The approach is focused on the relations between the combinatorics of graphs and algebraic properties of edge ideals. We refer the reader to see e.g. [3–5, 8–10, 12, 52, 63, 75, 89, 96, 112, 115, 151] for more information on this topic.

In Chapter 6 and Chapter 7, partly inspired by the interest of this problem, we study the regularity of edge ideals and their powers for the families of bipartite graphs and bicyclic graphs, respectively.

Bipartite graphs

In Chapter 6, we consider several aspects of the Rees algebra of the edge ideal of a bipartite graph.

One can find a vast literature on the Rees algebra of edge ideals of bipartite graphs (see e.g. [54, 56, 136, 147–150]), nevertheless, in Chapter 6 we study several properties that might have been overlooked. From a computational point of view we first focus on the universal Gröbner basis of its defining equations, and from a more algebraic standpoint we focus on its total and partial regularities as a bigraded algebra. Applying these ideas, we give an estimation of when $\mathrm{reg}(I^s)$ starts to be a linear function and we find upper bounds for the regularity of the powers of I.

Inside this subsection, let $G = (V(G), E(G))$ be a bipartite graph on the vertex set $V(G) = \{x_1, \dots, x_r\}$. As before, let $\mathbb{k}$ be a field, R be the polynomial $R = \mathbb{k}[x_1, \dots, x_r]$, and I be the edge ideal $I = I(G)$ of G. Let $f_1, \dots, f_q$ be the square free monomials of degree two generating I. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, \dots, T_q]$ via the map

$$S = \mathbb{k}[x_1, \dots, x_r, T_1, \dots, T_q] \xrightarrow{\psi} \mathcal{R}(I) \subset R[t], \quad \psi(T_i) = f_i t.$$

Then, the presentation of $\mathcal{R}(I)$ is given by $S/\mathcal{K}$ where $\mathcal{K} = \mathrm{Ker}(\psi)$.

The universal Gröbner basis of the ideal $\mathcal{K}$ is defined as the union of all the reduced Gröbner bases $\mathcal{G}_<$ of the ideal $\mathcal{K}$ as $<$ runs over all possible monomial orders (see [138]). In the first main result of Chapter 6, we compute the universal Gröbner basis of the defining equations $\mathcal{K}$ of the Rees algebra $\mathcal{R}(I)$.

Notation P (Notation 6.3)**.** *Given a walk $w = \{v_0, \dots, v_a\}$, each edge $\{v_{j-1}, v_j\}$ corresponds to a variable T_{i_j}, and we set $T_{w^+} = \prod_{j \text{ is even}} T_{i_j}$ and $T_{w^-} = \prod_{j \text{ is odd}} T_{i_j}$ (in case $a = 1$ we make $T_{w^+} = 1$). We adopt the following notations:*

(i) Let $w = \{v_0, \dots, v_a = v_0\}$ be an even cycle in G. Then, by T_w we denote the binomial

$$T_{w^+} - T_{w^-} \in \mathcal{K}.$$

(ii) Let $w = \{v_0, \dots, v_a\}$ be an even path in G. Then, the path w determines the binomial

$$v_0 T_{w^+} - v_a T_{w^-} \in \mathcal{K}.$$

(iii) Let $w_1 = \{u_0, \dots, u_a\}$, $w_2 = \{v_0, \dots, v_b\}$ be two disjoint odd paths. Let $T_{(w_1,w_2)^+} = T_{w_1^+} T_{w_2^-}$ and $T_{(w_1,w_2)^-} = T_{w_1^-} T_{w_2^+}$, then w_1 and w_2 determine the binomial

$$u_0 u_a T_{(w_1,w_2)^+} - v_0 v_b T_{(w_1,w_2)^-} \in \mathcal{K}.$$

Theorem Q (Theorem 6.5)**.** *Let G be a bipartite graph and $\mathcal{K}$ be the ideal of defining equations of the Rees algebra $\mathcal{R}(I(G))$. The universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$ is given by*

$$\begin{aligned} \mathcal{U} = \{&T_w \mid w \text{ is an even cycle}\} \\ &\cup \{v_0 T_{w^+} - v_a T_{w^-} \mid w = (v_0, \dots, v_a) \text{ is an even path}\} \\ &\cup \{u_0 u_a T_{(w_1,w_2)^+} - v_0 v_b T_{(w_1,w_2)^-} \mid w_1 = (u_0, \dots, u_a) \text{ and} \\ &\qquad\qquad\qquad\qquad w_2 = (v_0, \dots, v_b) \text{ are disjoint odd paths}\}. \end{aligned}$$

The polynomial ring S is equipped with the bigrading: $\text{bideg}(x_i) = (1, 0)$ and $\text{bideg}(T_j) = (0, 1)$. The algebra $\mathcal{R}(I)$, as a bigraded S-module, has a minimal bigraded free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{R}(I) \longrightarrow 0,$$

where $F_i = \oplus_j S(-a_{ij}, -b_{ij})$. In the same way as in [122], we can define the partial x-regularity of $\mathcal{R}(I)$ by

$$\text{reg}_{\mathbf{x}}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} - i\},$$

the partial T-regularity by

$$\text{reg}_{\mathbf{T}}(\mathcal{R}(I)) = \max_{i,j}\{b_{ij} - i\},$$

and the total regularity by

$$\text{reg}\,(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} + b_{ij} - i\}.$$

In the second main result of Chapter 6, we prove that the total regularity of $\mathcal{R}(I(G))$ coincides with the matching number of G and estimate both partial regularities of $\mathcal{R}(I(G))$.

Theorem R (Theorem 6.22)**.** *Let* G *be a bipartite graph. Then, we have:*

(i) $\operatorname{reg}(\mathcal{R}(I(G))) = \operatorname{match}(G)$,

(ii) $\operatorname{reg}_{\mathbf{x}}(\mathcal{R}(I(G))) \leqslant \operatorname{match}(G) - 1$,

(iii) $\operatorname{reg}_{\mathbf{T}}(\mathcal{R}(I(G))) \leqslant \operatorname{match}(G)$,

where $\operatorname{match}(G)$ *denotes the matching number of* G.

In Corollary 6.23, from the Chardin-Römer equality [27, Theorem 3.5], we obtain the upper bound

$$\operatorname{reg}(I(G)^s) \;\leqslant\; 2s + \operatorname{reg}_{\mathbf{x}}(\mathcal{R}(I)) \;\leqslant\; 2s + \operatorname{match}(G) - 1$$

for any bipartite graph G (this upper bound in some cases is weaker than the one of [90], but it is interesting how it follows from a mostly algebraic method). In Corollary 6.24, by using the relation between the partial T-regularity and the stabilization of the regularity of the powers of I [42, Proposition 3.7], we obtain that

$$\operatorname{reg}(I(G)^{s+1}) = \operatorname{reg}(I(G)^s) + 2.$$

for all $s \geqslant \operatorname{match}(G) + q + 1$.

The basic outline of Chapter 6 is as follows. In Section 6.1, we compute the universal Gröbner basis of $\mathcal{K}$ (Theorem Q). In Section 6.2, we consider a specific monomial order that allows us to get upper bounds for the partial x-regularity of $\mathcal{R}(I)$. In Section 6.3, we exploit the canonical module of $\mathcal{R}(I)$ in order to prove Theorem R. In Section 6.4, we give some general ideas about a conjectured upper bound (Conjecture 6.9).

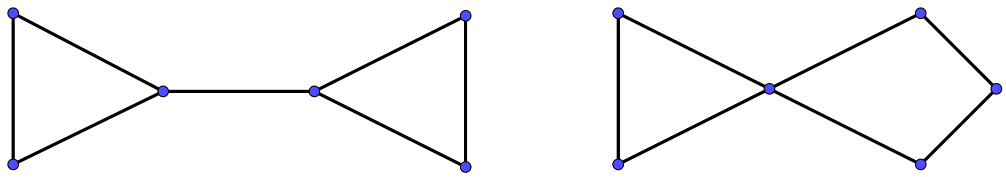
Bicyclic graphs

In Chapter 7, we study the regularity of the edge ideal and its powers for the case of bicyclic graphs.

A bicyclic graph is a graph containing exactly two cycles. The family of bicyclic graphs has three possible types of base graphs: two cycles joined at a vertex, two cycles connected by a path, and two cycles sharing a path. In [61], it was computed the regularity of all the powers of the edge ideal of a graph given by two cycles joined at a vertex.

In Chapter 7, we consider the family of bicyclic graphs where the base graph is a dumbbell. A dumbbell graph $C_n \cdot P_l \cdot C_m$ is a graph consisting of two cycles C_n and C_m connected with a path P_l, where n, m, and l are the number of vertices. In the example below we show two different dumbbell graphs.

Example S. *Two base cases when* $l = 2$ *and* $l = 1$ *are the following:*



The dumbbell graphs $C_3 \cdot P_2 \cdot C_3$ and $C_3 \cdot P_1 \cdot C_4$.

Notation T. *Let* G *be a graph and* $H \subset G$ *be a subgraph.*

(i) The maximum size of an induced matching in G *is called its induced matching number and it is denoted by* $\nu(G)$.

(ii) $\Gamma_G(H)$ *denotes the subset of vertices*

$$\Gamma_G(H) = \{v \in G \mid d(v, H) = 1\},$$

i.e. the vertices at distance one from H.

(iii) The Lozin transformation of a vertex $x \in G$ *is an operation that replaces* x *by four new vertices, and produces a new graph denoted by* $\mathcal{L}_x(G)$ *(see Definition 7.23).*

From [96, Corollary 1.2] and [11, Theorem 4.11] we have the following inequalities relating $\operatorname{reg} I(G)$ and $\nu(G)$ in the case of bicyclic graphs

$$\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 3.$$

In the main result of Chapter 7, we obtain a full combinatorial characterization of the regularity of the edge ideal of a bicyclic graph G (having a dumbbell subgraph) in terms of the induced matching number of G. This result adds a new family of graphs for which the regularity is known, and extends the results of [4] where the family of unicyclic graphs was studied. For a dumbbell graph $C_n \cdot P_l \cdot C_m$, we always assume that "n mod 3 $\leqslant$ m mod 3". Since the graphs $C_n \cdot P_l \cdot C_m$ and $C_m \cdot P_l \cdot C_n$ are clearly isomorphic, the cases "$n \equiv 2 \pmod 3$, $m \equiv 0, 1 \pmod 3$" have the same results as the cases "$n \equiv 0, 1 \pmod 3$, $m \equiv 2 \pmod 3$".

Theorem U (Theorem 7.64)**.** *Let* G *be a bicyclic graph with dumbbell subgraph* $C_n \cdot P_l \cdot C_m$.

(I) If $n, m \equiv 0, 1 \pmod 3$*, then* $\operatorname{reg} I(G) = \nu(G) + 1$.

(II) If $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$*, then*

$$\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 2,$$

and $\operatorname{reg} I(G) = \nu(G) + 2$ *if and only if* $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

(III) *If* $n, m \equiv 2 \pmod 3$ *and* $l \geqslant 3$*, then* $\nu(G) + 1 \leqslant \operatorname{regI}(G) \leqslant \nu(G) + 3$*. Moreover:*

(i) $\operatorname{regI}(G) = \nu(G) + 3$ *if and only if* $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$.

(ii) $\operatorname{regI}(G) = \nu(G) + 1$ *if and only if the following conditions hold:*

(a) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$*;*

(b) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$*;*

(c) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

(IV) *If* $n, m \equiv 2 \pmod 3$ *and* $l \leqslant 2$*, then* $\nu(G) + 1 \leqslant \operatorname{regI}(G) \leqslant \nu(G) + 2$*. Let* x *be a point on the bridge* P_l *and let* $\mathcal{L}_x(G)$ *be the Lozin transformation of* G *with respect to* x *given as in Construction 7.62. Then,* $\operatorname{regI}(G) = \nu(G) + 1$ *if and only if the following conditions are satisfied:*

(a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$*;*

(b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$*;*

(c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

For a particular family of dumbbell graphs, we obtain a formula for the regularity of all the powers of the edge ideal.

Theorem V (Theorem 7.76)**.** *For the dumbbell graph* $C_n \cdot P_l \cdot C_m$ *with* $l \leqslant 2$*, we have*

$$\operatorname{regI}(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{regI}(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geqslant 1$.

The basic outline of Chapter 7 is as follows. In Section 7.1, we recall some preliminary results. In Section 7.2, we compute the induced matching number of a dumbbell graph and the regularity of its edge ideal. In Section 7.3, we prove Theorem U. In Section 7.4, we prove Theorem V.

Note of references

Some parts of this work have already been submitted, published or accepted for publication in:

(1) ([29]) Yairon Cid-Ruiz, *A* D*-module approach on the equations of the Rees algebra*, to appear in J. Commut. Algebra (2017). arXiv:1706.06215.

(2) ([22]) Laurent Busé, Yairon Cid-Ruiz, and Carlos D'Andrea, *Degree and birationality of multi-graded rational maps*, ArXiv e-prints (May 2018), available at 1805.05180.

(3) ([30]) Yairon Cid-Ruiz, *Multiplicity of the saturated special fiber ring of height two perfect ideals*, ArXiv e-prints (July 2018). 1807.03189.

(4) ([33]) Yairon Cid-Ruiz and Aron Simis, *Degree of rational maps via specialization*, arXiv preprint arXiv:1901.06599 (2019).

(5) ([31]) Yairon Cid-Ruiz, *Regularity and Gröbner bases of the Rees algebra of edge ideals of bipartite graphs*, Le Matematiche 73 (2018), no. 2, 279–296.

(6) ([32]) Yairon Cid-Ruiz, Sepehr Jafari, Navid Nemati, and Beatrice Picone, *Regularity of bicyclic graphs and their powers*, to appear in J. Algebra Appl. (2018). arXiv:1802.07202.

A *Macaulay2* [60] package that gives support to most of the results of [22] was implemented. This package is capable of computing the saturated special fiber ring (the algebra introduced in [22]) of an ideal in a multi-graded setting. In particular, it provides new algorithms to compute the degree of a rational map and to test birationality. This package is included in the latest versions of *Macaulay2* [60]:

(7) ([28]) Yairon Cid-Ruiz, *MultiGradedRationalMap: Degree and birationality of multi-graded rational maps*. Available at `http://www2.macaulay2.com/Macaulay2/doc/Macaulay2-1.13/share/doc/Macaulay2/MultiGradedRationalMap/html/index.html`.

Chapter 1

Preliminaries

In this chapter we collect some basic definitions, concepts and results that will be used throughout this work. We are mostly interested in recalling some fundamental and known results in the *"theory of blow-up algebras"*.

For a very comprehensive treatment about blow-up algebras, the ultimate references are the books [144] and [146] by Vasconcelos.

1.1 The Rees algebra of an ideal

This initial section closely follows [83, Chapter 5] and [111].

For the time being, let R be a commutative ring with unit.

Definition 1.1. *Let $I \subset R$ be an ideal and t a variable over R. The **Rees algebra of** I is the subring of $R[t]$ defined as*

$$R[It] = \left\{ \sum_{i=0}^{n} a_i t^i \mid n \in \mathbb{N}, a_i \in I^i \right\} = \bigoplus_{i \geqslant 0} I^i t^i,$$

*and the **extended Rees algebra of I** is the subring of $R[t, t^{-1}]$ defined as*

$$R[It, t^{-1}] = \left\{ \sum_{i=-n}^{n} a_i t^i \mid n \in \mathbb{N}, a_i \in I^i \right\} = \bigoplus_{i \in \mathbb{Z}} I^i t^i,$$

where, by convention, for any non-positive integer n, $I^n = R$.

For every ideal J of R we have

$$J = JR[It] \cap R = JR[It, t^{-1}] \cap R = JR[t, t^{-1}] \cap R.$$

Thus every ideal of R is contracted from an ideal of $R[It]$ and $R[It,t^{-1}]$. Also

$$\frac{R}{J} \subset \frac{R[It]}{JR[t,t^{-1}] \cap R[It]} \subset \frac{R[It,t^{-1}]}{JR[t,t^{-1}] \cap R[It,t^{-1}]} \subset \frac{R[t,t^{-1}]}{JR[t,t^{-1}]},$$

where the two quotients in the middle are isomorphic to the Rees algebra and the extended Rees algebra of $(I+J)/J$ in the ring R/J. In particular, if P is a minimal prime ideal of R, then $PR[t,t^{-1}] \cap R[It]$ and $PR[t,t^{-1}] \cap R[It,t^{-1}]$ are minimal prime ideals in their respective rings. Any nilpotent element in $R[It]$ or $R[It,t^{-1}]$ is also nilpotent in $R[t,t^{-1}]$, so it lies in $\bigcap_P PR[t,t^{-1}]$, as P varies over the minimal primes of R. Hence all minimal primes of the two Rees algebras are contracted from minimal primes of $R[t,t^{-1}]$ each of which is of the form $PR[t,t^{-1}]$, for some minimal prime P of R. Therefore

$$\dim(R[It]) = \max\left\{ \dim\left(\frac{R}{P}\left[\frac{I+P}{P}t\right]\right) \mid P \in \operatorname{Min}(R)\right\}, \tag{1.1}$$

$$\dim(R[It,t^{-1}]) = \max\left\{ \dim\left(\frac{R}{P}\left[\frac{I+P}{P}t, t^{-1}\right]\right) \mid P \in \operatorname{Min}(R)\right\}. \tag{1.2}$$

Theorem 1.2. *Let R be a Noetherian ring and I an ideal of R. Then, $\dim R$ is finite if and only if the dimension of either the Rees algebra or the extended Rees algebra is finite. If $\dim R$ is finite, then*

(i) $\dim(R[It]) = \begin{cases} \dim(R)+1 & \text{if } I \not\subset P \text{ for some prime ideal} \\ & \text{with } \dim(R/P) = \dim(R); \\ \dim(R) & \text{otherwise.} \end{cases}$

(ii) $\dim(R[It,t^{-1}]) = \dim(R)+1$.

(iii) If R is local with maximal ideal $\mathfrak{m}$, and if $I \subseteq \mathfrak{m}$, then $\mathfrak{m}R[It,t^{-1}] + ItR[It,t^{-1}] + t^{-1}R[It,t^{-1}]$ is a maximal ideal in $R[It,t^{-1}]$ of height $\dim(R)+1$.

Proof. (i) From (1.1) we may assume that R is an integral domain, and so it is enough to prove that $\dim(R[It]) = \dim(R)$ if $I = 0$ (which is clear) and $\dim(R[It]) = 1 + \dim(R)$ if $I \neq 0$. From Lemma 1.24 we have that

$$\dim(R[It]) = \dim(R) + \operatorname{ht}(ItR[It]) = \dim(R[It]) + \operatorname{trdeg}_R(R[It]) = \dim(R) + 1.$$

(ii) Similarly, by (1.2) we may assume that R is an integral domain. From the Dimension Inequality [110, Theorem 15.5], we obtain $\dim(R[It,t^{-1}]) \leqslant 1 + \dim(R)$. On the other hand, we have $\dim\left(R[It,t^{-1}]\right) \geqslant \dim\left(R[It,t^{-1}]_{t^{-1}}\right) = \dim\left(R\left[It,t^{-1},\frac{1}{t^{-1}}\right]\right) = \dim\left(R[t,t^{-1}]\right) = \dim(R)+1$.

(iii) Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_h = \mathfrak{m}$ be a maximal chain of prime ideals in R with $h = \operatorname{ht}(\mathfrak{m}) = \dim(R)$. Set $Q_i = P_iR[t,t^{-1}] \cap R[It,t^{-1}]$. Since $Q_i \cap R = P_i$, $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_h$ is a chain of distinct prime ideals in $R[It,t^{-1}]$. The biggest one is $Q_h = \mathfrak{m}R[t,t^{-1}] \cap R[It,t^{-1}] =$

$\mathfrak{m}R[It, t^{-1}] + ItR[It, t^{-1}]$, which is properly contained in $Q_h + t^{-1}R[It, t^{-1}] = \mathfrak{m}R[It, t^{-1}] + ItR[It, t^{-1}] + t^{-1}R[It, t^{-1}]$. So the result follows. $\square$

Notation 1.3. *The Rees algebra* $R[It]$ *of the ideal* I*, will also be denoted by* $\mathcal{R}(I)$*. Sometimes, when we want to stress the role of the ring* R*, we will write* $\mathcal{R}_R(I)$*.*

The study of Rees algebras is strongly interrelated with the following two rings:

Definition 1.4. *The* ***associated graded ring*** *of* I *is defined by*

$$\mathrm{gr}_I(R) = \bigoplus_{n \geqslant 0} (I^n/I^{n+1}) = R[It]/IR[It] = R[It, t^{-1}]/t^{-1}R[It, t^{-1}].$$

If R *is Noetherian local with maximal ideal* $\mathfrak{m}$*, the* ***fiber cone*** *(or* ***special fiber ring****) of* I *is given by the ring*

$$\mathfrak{F}_R(I) = \frac{R[It]}{\mathfrak{m}R[It]} \cong \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \frac{I^3}{\mathfrak{m}I^3} \oplus \cdots .$$

The ***analytic spread*** *of* I*, denoted by* $\ell(I)$*, is equal to the dimension* $\ell(I) = \dim(\mathfrak{F}_R(I))$ *of* $\mathfrak{F}_R(I)$*.*

Clearly, we also have that $\mathfrak{F}_R(I) = \mathrm{gr}_I(R)/\mathfrak{m}\mathrm{gr}_I(R)$*.*

We recall the following well-known result: for a finitely generated M over a local ring $(R, \mathfrak{m})$, the *minimal number of generators* of M, denoted by $\mu(M)$, is equal to the dimension $\mu(M) = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$ of $M/\mathfrak{m}M$ as a vector space over $R/\mathfrak{m}$ (see e.g. [110, Theorem 2.3]).

Proposition 1.5. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* $I \subseteq \mathfrak{m}$ *an ideal. Then, the following statements hold:*

(i) $\dim(\mathrm{gr}_I(R)) = \dim(R)$.

(ii) $\ell(I) \leqslant \dim(R)$.

(iii) $\ell(IR_\mathfrak{p}) \leqslant \ell(I)$ *for all* $\mathfrak{p} \in V(I) \subset \mathrm{Spec}(R)$.

(iv) If I *is* $\mathfrak{m}$*-primary, then* $\ell(I) = \dim(R)$.

(v) $\mathrm{ht}(I) \leqslant \ell(I) \leqslant \mu(I)$.

Proof. (i) As t^{-1} is a non-zerodivisor in $R[It, t^{-1}]$ and $\mathrm{gr}_I(R) = R[It, t^{-1}]/t^{-1}R[It, t^{-1}]$, Theorem 1.2$(ii)$ implies that $\dim(\mathrm{gr}_I(R)) \leqslant \dim(R)$. Let $Q = \mathfrak{m}R[It, t^{-1}] + ItR[It, t^{-1}] + t^{-1}R[It, t^{-1}]$ be the prime ideal in Theorem 1.2(iii), since $\dim(\mathrm{gr}_I(R)) \geqslant \dim\left(\mathrm{gr}_I(R)_Q\right)$, it is enough to show $\dim\left(\mathrm{gr}_I(R)_Q\right) \geqslant R$. Now, t^{-1} is a non-zerodivisor in the local ring $R[It, t^{-1}]_Q$ and $\mathrm{gr}_I(R)_Q = R[It, t^{-1}]_Q/t^{-1}R[It, t^{-1}]_Q$, thus we obtain that $\dim\left(\mathrm{gr}_I(R)_Q\right) = \dim\left(R[It, t^{-1}]_Q\right) - 1 = \dim(R)$ (see e.g. [19, Proposition A.4], [110, Exercise 16.1]).

(ii) Since $\mathfrak{F}_R(I)$ is a quotient of $\mathrm{gr}_I(R)$, the result is clear from part (i).

(iii) Let $\mathfrak{p} \in V(I) \subset \mathrm{Spec}(R)$. The Hilbert functions of the graded algebras $\mathfrak{F}_R(I)$ and $\mathfrak{F}_{R_\mathfrak{p}}(IR_\mathfrak{p})$ are given by $H(\mathfrak{F}_R(I), n) = \dim_{R/\mathfrak{m}}(I^n/\mathfrak{m}I^n) = \mu(I^n)$ and $H\left(\mathfrak{F}_{R_\mathfrak{p}}(IR_\mathfrak{p}), n\right) = \dim_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}}(I^nR_\mathfrak{p}/\mathfrak{p}I^nR_\mathfrak{p}) = \mu(I^nR_\mathfrak{p})$. Clearly $\mu(I^n) \geqslant \mu(I^nR_\mathfrak{p})$, then $H(\mathfrak{F}_R(I), n) \geqslant H(\mathfrak{F}_{R_\mathfrak{p}}(IR_\mathfrak{p}), n)$, which from the relation of the Hilbert-Samuel polynomial and the Krull dimension implies $\ell(I) = \dim(\mathfrak{F}_R(I)) \geqslant \dim(\mathfrak{F}_{R_\mathfrak{p}}(IR_\mathfrak{p})) = \ell(IR_\mathfrak{p})$ (see e.g. [19, Theorem 4.1.3]).

(iv) If I is $\mathfrak{m}$-primary (i.e., $\mathfrak{m}^c \subset I$ for some $c > 0$), then the ideal $\mathfrak{m}\mathrm{gr}_I(R)$ is nilpotent. Therefore, $\ell(I) = \dim(\mathrm{gr}_I(R)/\mathfrak{m}\mathrm{gr}_I(R)) = \dim(\mathrm{gr}_I(R))$, and so part (i) implies $\ell(I) = \dim(R)$.

(v) Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be a minimal prime of I with $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$, due to parts (iii) and (iv) we get

$$\mathrm{ht}(I) = \dim(R_\mathfrak{p}) = \ell(IR_\mathfrak{p}) \leqslant \ell(I).$$

Let $n = \mu(I)$ and $f_1, \ldots, f_n$ be a minimal set of generators of I. Set a variable T_i for each one of the generators, then we have the following canonical epimorphism

$$\begin{aligned}(R/\mathfrak{m})[T_1, \ldots, T_n] &\twoheadrightarrow \mathfrak{F}_R(I) = \bigoplus_{m=0}^{\infty} I^m/\mathfrak{m}I^m\\ T_i &\mapsto \overline{f_i}.\end{aligned}$$

Thus it follows that $\ell(I) = \dim(\mathfrak{F}_R(I)) \leqslant \mu(I)$. ☐

Let $I = (f_1, \ldots, f_n) \subset R$ be an ideal. The Rees algebra $\mathcal{R}_R(I) = R[It] = R[f_1t, \ldots, f_nt]$ can be written as a homomorphic image of the polynomial ring $R[T_1, \ldots, T_n]$ by the map π sending T_i to f_it. The kernel is a graded ideal $J = J_1 + J_2 + \cdots \subset R[T_1, \ldots, T_n]$. Here we give cases where we can describe the generators of J, which we refer to as the **defining equations of the Rees algebra**.

The ideal J is generated by homogeneous polynomials $F(T_1, \ldots, T_n) \in R[T_1, \ldots, T_n]$ with the property $F(f_1t, \ldots, f_nt) = 0$. Since F is homogeneous, $F(f_1t, \ldots, f_nt) = 0$ if and only if $F(f_1, \ldots, f_n) = 0$. In the short exact sequence

$$0 \to J \to R[T_1, \ldots, T_n] \to \mathcal{R}_R(I) \to 0,$$

we are interested in finding the ideal J.

Generators for the linear part J_1 can be found with a presentation of I. Namely, given a presentation $R^m \xrightarrow{\varphi} R^n \to I \to 0$, where $\varphi = (a_{ij})$ represents the map $R^m \to R^n$, then the linear part J_1 is generated by the linear polynomials

$$g_i = a_{1i}T_1 + \ldots + a_{ni}T_n,$$

that is $J_1 = (g_1, \ldots, g_m)$.

From these previous discussions, we identify the symmetric algebra of I with

$$\mathrm{Sym}_R(I) \cong R[T_1, \ldots, T_n]/J_1.$$

Also we identify the Rees algebra of I with

$$\mathcal{R}_R(I) \cong R[T_1, \dots, T_n]/J.$$

So we have the following commutative diagram

$$\begin{tikzcd} R[T_1, \dots, T_n] \arrow[d] \arrow[dr] & \\ \mathrm{Sym}_R(I) \arrow[r, two heads, "\alpha"] & \mathcal{R}_R(I), \end{tikzcd}$$

where α is an epimorphism.

Definition 1.6. *An ideal* I *is said to be of* ***linear type*** *if* $J = J_1$ *(i.e.,* $\mathrm{Sym}_R(I) \cong \mathcal{R}_R(I)$*).*

Ideals of linear type are the simplest in terms of the defining equations of their Rees algebras. In fact, the symmetric algebra can always be computed from a presentation of the ideal. If R is an integral domain then clearly $\mathcal{R}_R(I)$ is also an integral domain.

The following important result was first obtained in [111].

Proposition 1.7. *Let* R *be an integral domain and* I *an ideal of* R*. The following conditions are equivalent:*

(i) $\mathrm{Sym}_R(I)$ *is an integral domain;*

(ii) $\mathrm{Sym}_R(I)$ *has no* R*-torsion;*

(iii) α *is injective* (*and hence* $\mathrm{Sym}_R(I) \cong \mathcal{R}_R(I)$).

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are trivial. The implication $(ii) \Rightarrow (iii)$ follows from Lemma 1.8(ii) below. $\square$

Lemma 1.8. *Let* $I = (f_1, \dots, f_n)$ *be an ideal of* R *and*

$$\mathcal{K} = \mathrm{Ker}\big(\alpha : \mathrm{Sym}_R(I) \to \mathcal{R}_R(I)\big) \cong J/J_1.$$

Then, the following statements hold:

(i) There exists an integer $c > 0$ *such that* $f_j^c \cdot \mathcal{K} = 0$ *for all* $1 \leqslant j \leqslant n$*.*

(ii) Additionally, if R *is an integral domain, then the kernel* $\mathcal{K}$ *is equal to the* R*-torsion submodule of* $\mathrm{Sym}_R(I)$*.*

Proof. (i) We can assume that $j = n$. We shall prove that for every homogeneous polynomial $F \in R[T_1, \dots, T_n]$ with $F(f_1, \dots, f_n) = 0$ there exists some $c > 0$ such that $f_n^c F \in J_1$.

Let $F \in J \subset R[T_1, \dots, T_n]$. We proceed by induction on the degree of the polynomial F; for degree 1 the result is vacuous. So, we assume that F has degree $d > 1$ and we write it as

$$F = T_1 H_1(T_1, \dots, T_n) + T_2 H_2(T_2, \dots, T_n) + \cdots + T_n H_n(T_n),$$

where each $H_i \in R[T_i, \dots, T_n] \subset R[T_1, \dots, T_n]$ is a homogeneous polynomial of degree $d-1$.

We define the degree 1 homogeneous polynomial

$$G = T_1 H_1(f_1, \dots, f_n) + T_2 H_2(f_2, \dots, f_n) + \cdots + T_n H_n(f_n) \in R[T_1, \dots, T_n],$$

that belongs to J_1. We write

$$\begin{aligned} f_n^{d-1} F - T_n^{d-1} G = T_1 &\Big(f_n^{d-1} H_1(T_1, \dots, T_n) - T_n^{d-1} H_1(f_1, \dots, f_n)\Big) \\ &+ T_2 \Big(f_n^{d-1} H_2(T_2, \dots, T_n) - T_n^{d-1} H_2(f_2, \dots, f_n)\Big) \\ &\cdots + T_n \Big(f_n^{d-1} H_n(T_n) - T_n^{d-1} H_n(f_n)\Big) \\ = {} & T_1 G_1(T_1, \dots, T_n) + \cdots + T_n G_n(T_n), \end{aligned}$$

where each G_i is a homogeneous $d-1$ degree polynomial in J. Using the inductive hypothesis there exist c_i such that $f_n^{c_i} G_i \in J_1$ for each $i = 1, \dots, n$. So we get $f_n^{d-1+c_1+\cdots+c_n} F \in J_1$, that concludes the proof of this part.

(ii) Clearly we have that the R-torsion of $\mathrm{Sym}_R(I)$ is contained in $\mathcal{K}$. In the other direction, since R is an integral domain, each $f_j \neq 0$ is a regular element and so part (i) implies that $\mathcal{K}$ is contained in the R-torsion of $\mathrm{Sym}_R(I)$. □

Definition 1.9. *Let R be a Noetherian ring and M be a finitely generated R-module. We say that M has rank r if one of the following equivalent conditions is satisfied:*

(i) $M \otimes_R K$ *is a free K-module of rank r where K denotes the total ring of fractions of R.*

(ii) $M_\mathfrak{p}$ *is a free $R_\mathfrak{p}$-module of rank r for each associated prime $\mathfrak{p} \in \mathrm{Ass}(R)$.*

(see [19, §1.4] *for details.)*

Below we generalize Lemma 1.8.

Lemma 1.10. *Let R be a Noetherian ring and I be an ideal having rank. Then, we have*

$$\mathcal{R}_R(I) \cong \frac{\mathrm{Sym}_R(I)}{\mathcal{T}_R\left(\mathrm{Sym}_R(I)\right)} = \frac{\mathrm{Sym}(I)}{H_I^0(\mathrm{Sym}(I))},$$

where $\mathcal{T}_R\left(\mathrm{Sym}_R(I)\right)$ denotes the R-torsion of $\mathrm{Sym}_R(I)$.

In particular, if R *is local with maximal ideal* $\mathfrak{m}$ *and* I *is* $\mathfrak{m}$*-primary then we have*

$$\mathcal{R}_R(I) = \frac{\mathrm{Sym}(I)}{H^0_{\mathfrak{m}}(\mathrm{Sym}(I))}.$$

Proof. By the assumption that I has rank then we have $\mathrm{grade}(I) \geqslant 1$ (see e.g. [19, proof of Corollary 1.4.7]), and from the Unmixedness Theorem (see e.g. [19, Exercise 1.2.21], [95, Theorem 125]) we can assume that $I = (f_1, \dots, f_n)$ where each f_j is an R-regular element. Thus Lemma 1.8(i) yields that $\mathcal{K} \subseteq H^0_I(\mathrm{Sym}_R(I)) \subseteq \mathcal{T}_R(\mathrm{Sym}_R(I))$. Since we always have $\mathcal{T}_R(\mathrm{Sym}_R(I)) \subseteq \mathcal{K}$, then we are done. ☐

In the last part of this section we show that an ideal generated by a regular sequence is of linear type. More generally, every ideal generated by a d-sequence is of linear type.

Definition 1.11 ([83, Definition 5.5.2])**.** *Let* R *be a commutative ring. Set* $f_0 = 0$. *A sequence of elements* $f_1, \dots, f_n$ *is said to be a* ***d-sequence*** *if one (and hence both) of the following equivalent conditions hold:*

(i) $(f_0, f_1, \dots, f_i) : f_{i+1}f_j = (f_0, f_1, \dots, f_i) : f_j$ *for all* $0 \leqslant i \leqslant n-1$ *and for all* $j \geqslant i+1$*;*

(ii) $(f_0, f_1, \dots, f_i) : f_{i+1} \bigcap (f_1, \dots, f_n) = (f_1, \dots, f_i)$ *for all* $0 \leqslant i \leqslant n-1$.

Remark 1.12. *A regular sequence is a particular case of a d-sequence.*

Let $F \in R[T_1, \dots, T_n]$. We define the weight of F to be i if $F \in (T_1, \dots, T_i)$ but $F \notin (T_1, \dots, T_{i-1})$. We set the weight to be 0 if $F = 0$.

Theorem 1.13. *Let* $f_1, \dots, f_n$ *be a d-sequence in* R*, and* $I = (f_1, \dots, f_n)$*. If* $F(T_1, \dots, T_n) \in R[T_1, \dots, T_n]$ *is a form of degree* e *such that* $F(f_1, \dots, f_n) \in (f_1, \dots, f_j)$*, then there exists a form* $G(T_1, \dots, T_n)$ *of degree* e *and weight at most* j *such that* $F - G \in J_1$ *(again* J_1 *is the linear part of the defining equations of the Rees algebra).*

Proof. We use induction on e. Suppose that $e = 1$. Since $F(f_1, \dots, f_n) \in (f_1, \dots, f_j)$, we may write $F(f_1, \dots, f_n) = \sum_{i=1}^{j} r_i f_i$. Set $G(T_1, \dots, T_n) = \sum_{i=1}^{j} r_i T_i$. Clearly G has degree 1 and weight at most j. So $F - G \in J_1$, because F has degree 1 and $(F - G)(f_1, \dots, f_n) = 0$.

Now we assume $e > 1$ and we use induction on the weight of F. If the weight of F is at most j we take $G = F$. If not, set $F = T_k F_1 + F_2$, where the weight of F is k and the weight of F_2 is at most $k-1$, and both F_1 and F_2 are homogeneous. Note that $\deg(F_1) = e - 1$. We have that $F(f_1, \dots, f_n) = f_k F_1(f_1, \dots, f_n) + F_2(f_1, \dots, f_n) \in (f_1, \dots, f_j)$, and $F_2(f_1, \dots, f_n) \in (f_1, \dots, f_{k-1})$. Hence

$$F_1(f_1, \dots, f_n) \in \big((f_1, \dots, f_{k-1}) : f_k\big) \cap I = (f_1, \dots, f_{k-1}).$$

We apply induction on F_1 to obtain a homogeneous polynomial G_1 of degree $e - 1$ and weight at most $k - 1$ such that $F_1 - G_1 \in J_1$.

Set $G' = T_k G_1 + F_2$. The weight of G' is at most $k-1$. Moreover, $F - G' = T_k(F_1 - G_1)$. Notice that $G'(f_1, \dots, f_n) = F(f_1, \dots, f_n) \in (f_1, \dots, f_j)$ and G' has weight at most $k-1$ and degree e. By induction there exists a homogeneous polynomial G of degree e and weight at most j such that $G' - G \in J_1$. It follows that $F - G = (F - G') + (G' - G) \in J_1$, finishing the proof. □

Corollary 1.14. *If* $f_1, \dots, f_n$ *is a d-sequence in* R*, then the ideal* $I = (f_1, \dots, f_n)$ *is of linear type.*

Proof. If $F \in J$ is homogeneous of degree d, then since $F(f_1, \dots, f_n) = 0$, we can apply Theorem 1.13 with weight $j = 0$ to conclude that $F \in J_1$. □

Corollary 1.15. *Let* $f_1, \dots, f_n$ *be a regular sequence. Then, the defining ideal* $J \subset R[T_1, \dots, T_n]$ *of the Rees algebra of* $(f_1, \dots, f_n)$ *is generated by the* 2×2 *minors of the matrix*

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ T_1 & T_2 & \cdots & T_n \end{pmatrix}.$$

Proof. Due to Corollary 1.14, the Rees algebra of $(f_1, \dots, f_n)$ coincides with its symmetric algebra. Since $f_1, \dots, f_n$ is a regular sequence, the syzygies of $(f_1, \dots, f_n)$ are given by the Koszul syzygies $x_i T_j - x_j T_i$. □

1.2 The Rees algebra of a module

This section is devoted to digress in the possible definitions/generalizations of the Rees algebra for a module. Certainly we would like that this new definitions coincide in the case of an ideal with the previous definition $\mathcal{R}_R(I) = \oplus_{n=0}^{\infty} I^n t^n = R[It] \subset R[t]$. We present three possible definitions and say under which conditions they are equivalent.

One may think at first sight that defining the Rees algebra of a module is an useless generalization. But this is completely false, the notion of the Rees algebra of a module appears naturally in many important constructions and applications. For instance, in Section 3.3, to develop the Jacobian dual criterion in a multi-graded setting, the concept of Rees algebras for modules will be an essential tool.

Here we will use some basic properties of symmetric algebras, we refer the reader to [47, Appendix 2] for a detailed treatment.

Definition 1.16 (First definition for the Rees algebra of a module)**.** *Let* M *be an* R*-module and* $g : M \to R^n$ *be an injective homomorphism, then the Rees algebra of the module* M *is defined as the image of the map* $\mathrm{Sym}_R(g) : \mathrm{Sym}_R(M) \to \mathrm{Sym}_R(R^n) = R[T_1, \dots, T_n]$.

This first generalization is the most naive possible, and indeed for an ideal $I \subset R$ we have that $\mathcal{R}_R(I)$ is the image of the map $\mathrm{Sym}_R(I) \to \mathrm{Sym}_R(R) = R[t]$. But unfortunately this definition has major drawbacks because it is not an invariant of the module and it depends on the embedding used. In [50, Example 1.1], there is an example of an embedding $g : I \to R^2$ of an ideal I, where the algebra given by image of $\mathrm{Sym}(g) : \mathrm{Sym}_R(I) \to \mathrm{Sym}_R(R^2)$ is not isomorphic to $\mathcal{R}_R(I) = \oplus_{n=0}^{\infty} I^n t^n$.

Remark 1.17. *The second definition will be given trying to extend the relation between* $\mathrm{Sym}_R(I)$ *and* $\mathcal{R}_R(I)$ *over an integral domain that we found in Lemma 1.8.* ***This will be our preferred definition and we will always use it as default****.*

Definition 1.18 (Second (and **our default**) definition for the Rees algebra of a module [132])**.** *Let* R *be a Noetherian ring and* M *be a finitely generated* R*-module having a rank. The Rees algebra* $\mathcal{R}_R(M)$ *of* M *is defined to be* $\mathrm{Sym}_R(M)$ *modulo its* R*-torsion submodule.*

From Lemma 1.10, under the previous assumptions the second definition agrees with the usual definition for an ideal.

We recall the following well-known fact.

Definition-Lemma 1.19. *Let* R *be a Noetherian ring and* M *be an* R*-module (not necessarily finitely generated). We say that* M *is torsion if one of the following equivalent conditions is satisfied:*

(i) $M \otimes_R K = 0$ *where* K *denotes the total ring of fractions of* R.

(ii) $M_{\mathfrak{p}} = 0$ *for each associated prime* $\mathfrak{p} \in \mathrm{Ass}(R)$.

Proof. $(i) \Rightarrow (ii)$ This is clear because $M_{\mathfrak{p}}$ is a localization of $M \otimes_R K$.

$(ii) \Rightarrow (i)$ Let S be the multiplicative set of non-zerodivisors of R. We need to prove that for any $0 \neq m \in M$ we have $\mathrm{Ann}_R(m) \cap S \neq \emptyset$.

Assume by contradiction that $\mathrm{Ann}_R(m) \cap S = \emptyset$ for some $0 \neq m \in M$. Thus $\mathrm{Ann}_R(m) \subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$ where $\mathrm{Ass}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ (see e.g. [110, Theorems 6.1 and 6.5]), and by the prime avoidance lemma (see e.g. [7, Proposition 1.11]), $\mathrm{Ann}_R(m) \subseteq \mathfrak{p}_i$ for some i. Therefore, we obtain the contradiction $0 \neq R_{\mathfrak{p}_i} \cdot m \subset M_{\mathfrak{p}_i}$. So the proof follows. □

When M is a finitely generated torsion-free module having rank, then Definition 1.16 and Definition 1.18 agree:

Lemma 1.20. *Let* R *be a Noetherian ring and* M *be a finitely generated* R*-module. Then, the following statements hold:*

(i) If M *has rank and* $g : M \to R^n$ *is any embedding into a free* R*-module, then we have an isomorphism*

$$\frac{\mathrm{Sym}_R(M)}{\mathcal{T}_R\left(\mathrm{Sym}_R(M)\right)} \cong \mathrm{Im}\left(\mathrm{Sym}_R(g)\right),$$

where $\mathcal{T}_R\left(\mathrm{Sym}_R(M)\right)$ *denotes the* R*-torsion of* $\mathrm{Sym}_R(M)$ *and* $\mathrm{Im}\left(\mathrm{Sym}_R(g)\right)$ *denotes the image of the natural map* $\mathrm{Sym}_R(g) : \mathrm{Sym}_R(M) \to \mathrm{Sym}_R(R^n)$.

(ii) If M *is torsion-free and has rank* r*, then there exists an embedding* $g : M \to R^r$.

Proof. (i) (this part is essentially taken from [50, Theorem 1.6]) We need to prove that kernel of the homomorphism

$$\mathrm{Sym}_R(g) : \mathrm{Sym}_R(M) \to \mathrm{Sym}_R(R^n) = R[T_1, \ldots, T_n]$$

is exactly the R-torsion of $\mathrm{Sym}_R(M)$. Let K be the total ring of fractions of R. We always have that $\mathcal{T}_R(\mathrm{Sym}_R(M)) \subseteq \mathrm{Ker}(\mathrm{Sym}_R(g))$, so it is enough to show that $\mathrm{Ker}(\mathrm{Sym}_R(g)) \otimes_R K = 0$. Equivalently, from Definition-Lemma 1.19, we can show that $\mathrm{Ker}(\mathrm{Sym}_R(g)) \otimes_R R_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$.

Since the formation of symmetric algebras commutes with localization, we can localize at any associated prime and assume that R is local with maximal ideal $\mathfrak{m}$ and $\mathrm{depth}(R) = 0$. Since M has rank, then it is free, say $M = R^r$. After these reductions, $g : R^r \to R^n$ is injective and we need to prove that

$$\mathrm{Sym}_R(g) : \mathrm{Sym}_R(R^r) = R[U_1, \ldots, U_r] \to \mathrm{Sym}_R(R^n) = R[T_1, \ldots, T_n]$$

is injective. Since $\mathrm{depth}(R) = 0$, $\mathfrak{m} \in \mathrm{Ass}(R)$, and so [19, Lemma 1.3.4] yields that g splits and we have that $g(R^r)$ is a direct summand of R^n (i.e. $R^n \cong g(R^r) \oplus R^{n-r}$). Therefore, by choosing an appropriate basis on the target R^n of g, we obtain that $\mathrm{Sym}_R(g)$ corresponds with a canonical inclusion of the form $\mathrm{Sym}_R(R^r) = R[U_1, \ldots, U_r] \hookrightarrow R[U_1, \ldots, U_r, T'_{r+1}, \ldots, T'_n] \cong \mathrm{Sym}_R(R^n)$. So the proof of this part follows.

(ii) This is well-known (see e.g. [19, Exercise 1.4.18]). □

The last definition is the most complete in the sense of requiring less assumptions, but on the other hand it is more abstract and sometimes more difficult to handle.

Definition 1.21 (Third definition for the Rees algebra of a module [50])**.** *If R is a ring and M is an R-module, we define the Rees algebra of M to be*

$$\mathcal{R}_R(M) = \mathrm{Sym}_R(M)/\left(\cap_g L_g\right)$$

where the intersection is taken over all maps g from M to free R-modules, and L_g denotes the kernel of $\mathrm{Sym}_R(g)$.

Remark 1.22. *Let R be a Noetherian ring and M be a finitely generated R-module having rank, then Definition 1.18 and Definition 1.21 agree.*

Proof. See [50, paragraph after Lemma 1.7]. □

1.3 Krull dimension of symmetric and Rees algebras

The purpose of this section is to develop formulas to compute the Krull dimension of the symmetric and Rees algebras of a module.

First we start with the symmetric algebra, for this we introduce the Foster-Swan number.

Definition 1.23. *Let M be an R-module, then the Foster-Swan number is*

$$b(M) = \sup_{\mathfrak{p}\in\operatorname{Spec}(R)} \left\{ \dim(R/\mathfrak{p}) + \mu(M_{\mathfrak{p}})\right\},$$

where μ denotes minimal number of generators.

The following important lemma ([135, Lemma 1.1.2], [144, Lemma 1.2.2]) is popularly known as the "dimension formula for graded domains".

Lemma 1.24. *Let B be a Noetherian integral domain that is finitely generated over a subring A. Suppose there exists a prime ideal Q of B such that $B = A + Q$ and $A \cap Q = 0$. Then*

$$\dim(B) = \dim(A) + \operatorname{ht}(Q) = \dim(A) + \operatorname{trdeg}_A(B).$$

Proof. We may assume that $\dim(A)$ is finite. By our assumptions, we have $\dim(B) \geqslant \dim(A) + \operatorname{ht}(Q)$ (using the map $B \to B/Q = A$, we can construct an increasing sequence of prime ideals of length $\dim(A) + \operatorname{ht}(Q)$). On the other hand, we use the Dimension Inequality [110, Theorem 15.5] to obtain

$$\operatorname{ht}(P) \leqslant \operatorname{ht}(\mathfrak{p}) + \operatorname{trdeg}_A(B) - \operatorname{trdeg}_{k(\mathfrak{p})} k(P),$$

for any prime ideal $P \subset B$ and $\mathfrak{p} = P \cap A$. Thus we get

$$\dim(B) \leqslant \dim(A) + \operatorname{trdeg}_A(B).$$

Let $S = A - (0)$, $K = \operatorname{Quot}(A) = S^{-1}A$ and $B' = S^{-1}B$, then we may see B' as a finitely generated K-algebra and from [110, Theorem 5.6] we have

$$\operatorname{trdeg}_A(B) = \operatorname{trdeg}_K(B') = \dim(B').$$

Finally, since $A \cap Q = 0$ and $B = A + Q$, we get $\dim(B') = \operatorname{ht}(Q)$. $\square$

There are two cases of particular interest. If B is a Noetherian graded ring and A denotes its degree 0 component then:

$$\dim(B/P) = \dim(A/\mathfrak{p}) + \operatorname{trdeg}_{k(\mathfrak{p})}(k(P))$$

for any graded prime $P \in \operatorname{Spec}(R)$ and $\mathfrak{p} = P \cap A$, and

$$\dim(B_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}) + \operatorname{trdeg}_A(B)$$

for any $\mathfrak{p} \in \operatorname{Spec}(A)$.

In the next lemma we gather some basic facts when R is a Noetherian integral domain. Notice that in this case any finitely generated R-module has rank and so we can freely apply our default definition (Definition 1.18) for Rees algebras.

Lemma 1.25. *Let* R *be a Noetherian integral domain and* M *be a finitely generated* R*-module. Let* $K = \mathrm{Quot}(R)$ *be the field of fractions of* R*. Then, the following statements hold:*

(i) $\mathcal{R}_R(M)$ *is an integral domain and there is a canonical inclusion*

$$\mathcal{R}_R(M) = \frac{\mathrm{Sym}_R(M)}{\mathcal{T}_R\left(\mathrm{Sym}_R(M)\right)} \hookrightarrow \mathrm{Sym}_R(M) \otimes_R K \cong \mathrm{Sym}_K(M \otimes_R K) = K[T_1, \ldots, T_r]$$

where $r = \mathrm{rank}(M)$.

(ii) $\mathrm{trdeg}_R\left(\mathcal{R}_R(M)\right) = \mathrm{rank}(M)$.

(iii) $\dim\left(\mathcal{R}_R(M)\right) = \dim(R) + \mathrm{rank}(M)$.

Proof. (i) There is a canonical map $\gamma : \mathrm{Sym}_R(M) \to \mathrm{Sym}_R(M) \otimes_R K \cong \mathrm{Sym}_K(M \otimes_R K) = K[T_1, \ldots, T_r]$ where $r = \mathrm{rank}(M) = \dim_K(M \otimes_R K)$. The result follows because by construction $\mathrm{Ker}(\gamma) = \mathcal{T}_R\left(\mathrm{Sym}_R(M)\right)$.

(ii) It is clear form part (i) and the fact that

$$\mathcal{R}_R(M) \otimes_R K = \frac{\mathrm{Sym}_R(M)}{\mathcal{T}_R\left(\mathrm{Sym}_R(M)\right)} \otimes_R K \cong \mathrm{Sym}_R(M) \otimes_R K.$$

(iii) It is obtained from Lemma 1.24 and part (ii). □

Now we are ready for the Huneke-Rossi dimension formula of symmetric algebras ([81]).

Theorem 1.26. *Let* R *be a Noetherian ring and* M *be a finitely generated* R*-module. Then*

$$\dim\left(\mathrm{Sym}_R(M)\right) = b(M).$$

Proof. We make use of the observation that if R is an integral domain, then the R-torsion submodule of a symmetric algebra $\mathrm{Sym}_R(M)$ is a prime ideal of $\mathrm{Sym}_R(M)$.

For a prime ideal $\mathfrak{p}$ of R, we denote by $T(\mathfrak{p})$ the $R/\mathfrak{p}$-torsion submodule of

$$\mathrm{Sym}_R(M) \otimes_R R/\mathfrak{p} \cong \mathrm{Sym}_{R/\mathfrak{p}}(M/\mathfrak{p}M),$$

thus $\mathcal{R}_{R/\mathfrak{p}}\left(M/\mathfrak{p}M\right) \cong \left(\mathrm{Sym}_R(M) \otimes_R R/\mathfrak{p}\right)/T(\mathfrak{p})$. From Lemma 1.25 above, we have

$$\begin{aligned} \dim\left(\mathcal{R}_{R/\mathfrak{p}}\left(M/\mathfrak{p}M\right)\right) &= \dim(R/\mathfrak{p}) + \mathrm{rank}_{R/\mathfrak{p}}\left(M/\mathfrak{p}M\right) \\ &= \dim(R/\mathfrak{p}) + \mu(M_{\mathfrak{p}}), \end{aligned}$$

Since $\mathcal{R}_{R/\mathfrak{p}}\left(M/\mathfrak{p}M\right)$ can be seen as a quotient of $\mathrm{Sym}_R(M)$, it follows that $\dim\left(\mathrm{Sym}_R(M)\right) \geqslant b(M)$.

Conversely, let P be a prime of $\mathrm{Sym}_R(M)$ and set $\mathfrak{p} = P \cap R$. There is a canonical surjection $\mathrm{Sym}_R(M) \otimes_R R/\mathfrak{p} \twoheadrightarrow \mathrm{Sym}_R(M)/P$, and by taking out the $R/\mathfrak{p}$-torsion we obtain the surjection

$$\mathcal{R}_{R/\mathfrak{p}}(M/\mathfrak{p}M) \twoheadrightarrow \mathrm{Sym}_R(M)/P.$$

So it is clear that $\dim(\mathrm{Sym}_R(M)/P) \leqslant \dim\left(\mathcal{R}_{R/\mathfrak{p}}(M/\mathfrak{p}M)\right)$.

Therefore, it follows that $\dim(\mathrm{Sym}_R(M)) \leqslant b(M)$. □

Next we compute the Krull dimension of the Rees algebra of a module.

Theorem 1.27 ([132, Proposition 2.2])**.** *Let R be a Noetherian ring and M be a finitely generated R-module having rank. Then*

$$\dim(\mathcal{R}_R(M)) = \dim(R) + \mathrm{rank}(M).$$

Proof. Let $r = \mathrm{rank}(M)$. Since by the definition of the Rees algebra we mod out the R-torsion, we may assume that M is torsion-free. From Lemma 1.20(ii) we can embed M into a free module $G = R^r$. By using Lemma 1.20(i), $\mathcal{R}_R(M)$ is identified as a subalgebra of the polynomial ring $S = \mathrm{Sym}_R(G) = R[T_1, \ldots, T_r]$. As in the case of ideals, the minimal primes of $\mathcal{R}_R(M)$ are exactly of the form $P = \mathfrak{p}S \cap \mathcal{R}_R(M)$, where $\mathfrak{p}$ ranges over all minimal primes of R. Write $\overline{R} = R/\mathfrak{p}$ and $\overline{M}$ for the image of M in $\overline{R} \otimes_R G$. Since $\mathcal{R}_R(M)/P \cong \mathcal{R}_{\overline{R}}(\overline{M})$ and $\overline{M}$ has rank r as an $\overline{R}$-module, we may replace R and M by $\overline{R}$ and $\overline{M}$ to assume that R is an integral domain. But then the result follows from Lemma 1.25(iii). □

Remark 1.28. *If an ideal I has rank then it contains a regular element (*[19, proof of Corollary 1.4.7]*), which implies I cannot be contained in any minimal prime of R. Therefore, we have that both Theorem 1.2 and Theorem 1.27 agree with* $\dim(\mathcal{R}_R(I)) = 1 + \dim(R)$.

1.4 Certain Fitting conditions

In this section we review certain Fitting conditions that are important in the study of blow-up algebras. By imposing them one can deduce desirable properties (for instance, deducing that an ideal is of linear type or computing the analytic spread).

Let R be a commutative ring and M be a finitely generated R-module with finite presentation

$$R^m \xrightarrow{\varphi} R^n \to M \to 0. \tag{1.3}$$

The Fitting invariants of M are given by the various ideals generated by the minors of a matrix presentation of φ.

Definition 1.29. *Given an $n \times m$ matrix φ with entries in the commutative ring R, we set $I_t(\varphi)$ for the ideal generated by the $t \times t$ minors of the matrix φ. The ideal $I_1(\varphi)$ is called the content of φ. For systematic reasons we set $I_t(\varphi) = R$ for $t \leqslant 0$ and $I_t(\varphi) = 0$ for $t > \min(n, m)$.*

Definition 1.30. *For any integer* r*, the ideal* $\mathrm{Fitt}_r(M)$ *generated by the minors of order* $n-r$ *of the matrix* φ *is the* r*-th Fitting ideal of* M*, that is* $\mathrm{Fitt}_r(M) = I_{n-r}(\varphi)$.

Lemma 1.31. *The previous definition of* $\mathrm{Fitt}_r(M)$ *is independent of the presentation (1.3) chosen.*

Proof. See e.g. [47, §20.2], [137, Tag 07Z6]. □

One of the most important properties of Fitting ideals is that they serve as an obstruction for the number of generators of a module.

Proposition 1.32 ([47, Proposition 20.6])**.** *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be a finitely generated* R*-module. Then,* M *can be generated by* r *elements if and only if* $\mathrm{Fitt}_r(M) = R$.

Now, let R be a Noetherian ring and $I \subset R$ be an ideal.

Definition 1.33. *Let* $m \geqslant 0$ *be an integer.*

(G) *(one allows* $m = \infty$*)* I *satisfies the condition* G_m *if* $\mu(I_\mathfrak{p}) \leqslant \mathrm{ht}(\mathfrak{p})$ *for all* $\mathfrak{p} \in V(I) \subset \mathrm{Spec}(R)$ *such that* $\mathrm{ht}(\mathfrak{p}) < m$.

(F) I *satisfies the condition* F_m *if* $\mu(I_\mathfrak{p}) \leqslant \mathrm{ht}(\mathfrak{p}) + 1 - m$ *for all* $\mathfrak{p} \in V(I) \subset \mathrm{Spec}(R)$.

In terms of Fitting ideals we have the following translation:

Lemma 1.34. *Suppose* I *is an ideal of positive height. Then,* I *satisfies* G_m *if and only if* $\mathrm{ht}(\mathrm{Fitt}_i(I)) > i$ *for all* $1 \leqslant i < m$*, whereas* I *satisfies* F_m *if and only if* $\mathrm{ht}(\mathrm{Fitt}_i(I)) \geqslant m + i$ *for all* $i \geqslant 1$.

Proof. It follows from Proposition 1.32. □

These conditions were originally introduced in [6, Section 2, Definition] and [71, Lemma 8.2, Remark 8.3], respectively. Both conditions are more interesting when the cardinality of a global set of generators of I is large and m stays low. Thus, F_m is typically considered for $m = 0, 1$, while G_m gets its way when $m \leqslant \dim R$. Also, we have that F_1 is equivalent to G_∞.

First we note that G_∞ is an important necessary condition for an ideal being of linear type.

Lemma 1.35. *Let* R *be a Noetherian ring and* $I \subset R$ *be an ideal. Then, if* $\mathcal{R}(I) \cong \mathrm{Sym}(I)$ *then* I *satisfies the condition* G_∞.

Proof. See [71, Proposition 2.4]. □

An important family of ideals is the following:

Definition 1.36 ([82][144, Definition 3.3.9])**.** *Let* R *be a Noetherian local ring. An ideal* $I \subset R$ *is said to strongly Cohen-Macaulay, if the Koszul homology modules with respect to one (and then to any) generating set are Cohen-Macaulay.*

With this notion, we get a partial inverse to Lemma 1.35.

Theorem 1.37 ([69, Theorem 2.6]). *Let R be a Cohen-Macaulay ring and I be an ideal of positive grade. Assume that*

(i) I *satisfies the condition* G_∞.

(ii) I *is a strongly Cohen-Macaulay ideal.*

Then, I *is an ideal of linear type. Furthermore,* $\mathcal{R}(I)$ *is Cohen-Macaulay.*

Important families of strongly Cohen-Macaulay ideals are given by:

Proposition 1.38 ([82, Proposition 0.3][144, Corollary 4.2.5]). *Let R be a regular local ring and $I \subset R$ be an ideal. If I is either perfect of height two or Gorenstein of height three, then I is strongly Cohen-Macaulay.*

Combining these previous results we obtain a desirable equivalence:

Corollary 1.39. *Let R be a regular local ring and $I \subset R$ be an ideal. Assume that I is either perfect of height two or Gorenstein of height three. Then, the following conditions are equivalent:*

(i) I *satisfies the condition* G_∞.

(ii) I *is an ideal of linear type.*

Proof. It follows from Lemma 1.35, Theorem 1.37 and Proposition 1.38. □

We will use a version of Corollary 1.39 on the punctured spectrum of R.

Corollary 1.40. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and $I \subset R$ be an ideal. Assume that I is either perfect of height two or Gorenstein of height three. Then, the following conditions are equivalent:*

(i) I *satisfies the condition* G_d.

(ii) I *is an ideal of linear type on the punctured spectrum of* R *(i.e.,* $\mathrm{Sym}_{R_\mathfrak{p}}(I_\mathfrak{p}) \cong \mathcal{R}_{R_\mathfrak{p}}(I_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$*).*

Proof. Since the strongly Cohen-Macaulay condition localizes, we can assume that R is a regular local ring of dimension $\leqslant d-1$ and that I satisfies G_∞. Again, the result follows from Lemma 1.35, Theorem 1.37 and Proposition 1.38. □

Finally, we recall the following result for computing the analytic spread of an ideal.

Proposition 1.41 ([142, Corollary 4.3]). *Let R be a Cohen-Macaulay local ring of dimension d and let I be a strongly Cohen-Macaulay ideal of positive grade. Suppose that $\mu(I) \geqslant d+1$ and that I satisfies the G_d condition. Then $\ell(I) = d$.*

Part I

Algebra

Chapter 2

A D-module approach on the equations of the Rees algebra

Let $\mathbb{F}$ be a field of characteristic zero, $R = \mathbb{F}[x_1, x_2]$ be a polynomial ring in two variables, and $I = (f_1, f_2, f_3) \subset R$ be a height two ideal minimally generated by three homogeneous polynomials of the same degree d. The Rees algebra of I is defined as $\mathcal{R}(I) = R[It] = \bigoplus_{i=0}^{\infty} I^i t^i$. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, T_2, T_3]$ via the map

$$S = R[T_1, T_2, T_3] \xrightarrow{\psi} \mathcal{R}(I), \quad \psi(T_i) = f_i t. \tag{2.1}$$

In this chapter we are interested in the defining equations of the Rees algebra $\mathcal{R}(I)$, that is, the kernel $\mathcal{J} = \operatorname{Ker}(\psi)$ of this map ψ. The main feature of this chapter is the use of the theory of D-modules in the problem of finding the equations of $\mathcal{R}(I)$.

2.1 An "explicit" description of the equations

In this section we use the following setup.

Setup 2.1. *Let* $\mathbb{k}$ *be an arbitrary field, and* $R = \mathbb{k}[x_1, x_2]$ *be the polynomial ring in two variables. Let* $I \subset R$ *be a height two ideal minimally generated by three homogeneous polynomials* $\{f_1, f_2, f_3\}$ *of the same degree* d*. From the Hilbert-Burch Theorem we have a presentation*

$$0 \to R(-d-\mu) \oplus R(-2d+\mu) \xrightarrow{\varphi} R(-d)^3 \to I \to 0, \tag{2.2}$$

where the elements of the first column of φ *are homogeneous of degree* μ*, and the elements of the second column are homogeneous of degree* $d - \mu$*. Let* U *and* S *be the polynomial rings* $U = \mathbb{k}[T_1, T_2, T_3]$ *and* $S = R[T_1, T_2, T_3] = \mathbb{k}[x_1, x_2, T_1, T_2, T_3]$*, respectively. We regard* S *as a bigraded* $\mathbb{k}$*-algebra, where* $\operatorname{bideg}(T_i) = (1, 0)$ *and* $\operatorname{bideg}(x_i) = (0, 1)$*. The equations of the* *symmetric algebra are given by* $[g_1, g_2] = [T_1, T_2, T_3] \cdot \varphi$. *We are interested in the kernel* $\mathcal{K}$ *of the surjective map* $\alpha : \mathrm{Sym}(I) \to \mathcal{R}(I)$.

We denote by $S_{p,q}$ the $\mathbb{F}$-vector space spanned by the monomials $x_1^{\alpha_1} x_2^{\alpha_2} T_1^{\gamma_1} T_2^{\gamma_2} T_3^{\gamma_3}$ with $\gamma_1 + \gamma_2 + \gamma_3 = p$ and $\alpha_1 + \alpha_2 = q$. The map ψ from (2.1) becomes bihomogeneous when we declare $\mathrm{bideg}(t) = (1, -d)$, and also from the fact that $\mathrm{bideg}(g_1) = (1, \mu)$ and $\mathrm{bideg}(g_2) = (1, d - \mu)$, we get that $\mathcal{R}(I)$, $\mathrm{Sym}(I)$, $\mathcal{J}$ and $\mathcal{K}$ have natural structures as bigraded S-modules. For an arbitrary bigraded S-module N we use the notations

$$N_{p,*} = \bigoplus_{q \in \mathbb{Z}} N_{p,q} \qquad \text{and} \qquad N_{*,q} = \bigoplus_{p \in \mathbb{Z}} N_{p,q},$$

where $N_{p,*}$ is a graded R-module and $N_{*,q}$ is a graded U-module. For simplicity of notation, the R-module $N_{p,*}$ sometimes will be denoted by just N_p.

From Lemma 1.10 we can compute $\mathcal{K}$ as the torsion in $\mathrm{Sym}(I)$ with respect to the maximal ideal $\mathfrak{m} = (x_1, x_2) \subset R$, that is

$$\mathcal{K} = (0 :_{\mathrm{Sym}(I)} \mathfrak{m}^\infty) = H^0_{\mathfrak{m}}(\mathrm{Sym}(I)).$$

Given a bigraded S-module M, by definition each local cohomology module $H^j_{\mathfrak{m}}(M)$ is only an R-module. In the following lemma we endow $H^j_{\mathfrak{m}}(M)$ with a structure of bigraded S-module. We use [17, Chapter 13] for the foundations of local cohomology modules in the graded case.

Lemma 2.2. *Let* M *be a bigraded* S*-module. Then, the following statements hold:*

(i) Use the decomposition $M = \bigoplus_{p \in \mathbb{Z}} M_p$*, where* M_p *is the graded* R*-module given by* $M_p = \bigoplus_{q \in \mathbb{Z}} M_{p,q}$*. Then*

$$H^j_{\mathfrak{m}}(M) = \bigoplus_{p \in \mathbb{Z}} H^j_{\mathfrak{m}}(M_p), \tag{2.3}$$

is a bigraded S*-module with* $H^j_{\mathfrak{m}}(M)_{p,q} = H^j_{\mathfrak{m}}(M_p)_q$ *(where* $H^j_{\mathfrak{m}}(M_p)_q$ *represents the* q*-th graded part of the graded* R*-module* $H^j_{\mathfrak{m}}(M_p)$*). The actions of the* x_i*'s are natural because* $H^j_{\mathfrak{m}}(M)$ *is an* R*-module. The action of the* T_i*'s over* M *can be seen as homogeneous homomorphisms* $T_i : M_p \to M_{p+1}$ *of graded* R*-modules, then the induced homogeneous homomorphisms* $H^j_{\mathfrak{m}}(T_i) : H^j_{\mathfrak{m}}(M_p) \to H^j_{\mathfrak{m}}(M_{p+1})$ *of graded* R*-modules give us the action of the* T_i*'s over* $H^j_{\mathfrak{m}}(M)$.

(ii) For any $a, b \in \mathbb{Z}$*, we have the isomorphism of bigraded* S*-modules* $H^j_{\mathfrak{m}}(M(a,b)) \cong H^j_{\mathfrak{m}}(M)(a,b)$.

Proof. (i) The decomposition (2.3) comes from the fact that local cohomology commutes with direct sums, and that each $H^j_{\mathfrak{m}}(M_p)$ has a natural structure of graded R-module (see e.g. [17, Chapter 13]).

(ii) The shifting on the T_i's follows from the construction (2.3) and so we are left to check that $H^j_m(M_p(b)) \cong H^j_m(M_p)(b)$ for each $p \in \mathbb{Z}$. For this, we use [17, Theorem 13.4.5] and any of the remarks in page 273 of [17], for instance using the construction as a direct limit of Ext's we have

$$H^j_m(M_p(b)) \cong \varinjlim_n {}^*\mathrm{Ext}^j_R(R/m^n, M_p(b)) \cong \varinjlim_n {}^*\mathrm{Ext}^j_R(R/m^n, M_p)(b) \cong H^j_m(M_p)(b).$$

(see e.g. [19, Section 1.5] for graded dual ${}^*\mathrm{Hom}_R$ and its derived functors ${}^*\mathrm{Ext}^j_R$ in the category of graded modules). □

The "philosophy" that we follow in this section is similar to the one used in [108]. Explicitly, we shall try to find information by deleting the columns of φ and hopefully work with "simpler" modules. Let φ_1 be the matrix given by the first column of φ, then we we are interested in the module $E = \mathrm{Coker}(\varphi_1)$ with presentation

$$0 \to R(-d-\mu) \xrightarrow{\varphi_1} R(-d)^3 \to E \to 0.$$

Lemma 2.3. *For the module* E *we have*

(i) $\mathrm{Sym}(E) \cong S/(g_1)$*;*

(ii) $\mathrm{Sym}(E)$ *is an integral domain.*

Proof. (i) Follows from the presentation of E.

(ii) Since $I_1(\varphi_1) \supset I_2(\varphi)$, we have that $\mathrm{ht}(I_1(\varphi_1)) = 2$. Then, by [133, Theorem 3.4] we get that $\mathrm{Sym}(E)$ is an integral domain. □

Now we can find explicit relations between the local cohomology modules of $\mathrm{Sym}(I)$ and $\mathrm{Sym}(E)$ from the important fact that $\mathrm{Sym}(E)$ is an integral domain.

Lemma 2.4. *We have the following exact sequences of bigraded* S*-modules*

$$0 \to H^0_m(\mathrm{Sym}(I)) \xrightarrow{\partial} H^1_m\big(\mathrm{Sym}(E)\big)(-1, -d+\mu) \xrightarrow{g_2} H^1_m(\mathrm{Sym}(E)); \tag{2.4}$$

$$0 \to H^1_m(\mathrm{Sym}(E)) \xrightarrow{\partial} H^2_m\big(S\big)(-1, -\mu) \xrightarrow{g_1} H^2_m(S). \tag{2.5}$$

Proof. Since $\mathrm{Sym}(E) \cong S/(g_1)$ is an integral domain we have a short exact sequence

$$0 \to \mathrm{Sym}\big(E\big)(-1, -d+\mu) \xrightarrow{g_2} \mathrm{Sym}(E) \to \mathrm{Sym}(I) \to 0.$$

Using the corresponding long exact sequence in local cohomology and the fact that $H^0_m(\mathrm{Sym}(E)) = 0$, we get the required exact sequence

$$0 \to H^0_m(\mathrm{Sym}(I)) \xrightarrow{\partial} H^1_m\big(\mathrm{Sym}(E)\big)(-1, -d+\mu) \xrightarrow{g_2} H^1_m(\mathrm{Sym}(E)),$$

where ∂ is the induced connecting homomorphism.

Similarly, from the short exact sequence

$$0 \to S(-1,-\mu) \xrightarrow{g_1} S \to \mathrm{Sym}(E) \to 0, \tag{2.6}$$

and the fact that

$$H^j_{\mathfrak{m}}(R) \cong \begin{cases} x_1^{-1}x_2^{-1}\Bbbk[x_1^{-1},x_2^{-1}] & \text{if } j=2 \\ 0 & \text{otherwise,} \end{cases}$$

we can follow the same long exact sequence argument and obtain (2.5). $\square$

The next theorem contains the main result of this section, where we find an "explicit" way of computing the equations of the Rees algebra of I. We remark that this result is already known (see e.g. [39, Lemma 2.4] or [103, Theorem 2.4]), but we present a different proof. The rest of this chapter will depend on it.

Theorem 2.5. *Adopt Setup 2.1. Then, we have the following isomorphism of bigraded* S*-modules*

$$\mathcal{K} \cong \Big\{ w \in H^2_{\mathfrak{m}}(S)(-2,-d) \mid g_1 \cdot w = 0 \text{ and } g_2 \cdot w = 0 \Big\}.$$

Proof. The commutative diagram

$$\begin{tikzcd} S(-2,-d) \arrow[r, "g_1"] \arrow[d, "g_2"] & S(-1,-d+\mu) \arrow[d, "g_2"] \\ S(-1,-\mu) \arrow[r, "g_1"] & S \end{tikzcd}$$

can be extended to the following one with exact rows (each row is as in (2.6))

$$\begin{tikzcd} 0 \arrow[r] & S(-2,-d) \arrow[r, "g_1"] \arrow[d, "g_2"] & S(-1,-d+\mu) \arrow[r] \arrow[d, "g_2"] & \mathrm{Sym}(E)(-1,-d+\mu) \arrow[r] \arrow[d, "g_2"] & 0 \\ 0 \arrow[r] & S(-1,-\mu) \arrow[r, "g_1"] & S \arrow[r] & \mathrm{Sym}(E) \arrow[r] & 0. \end{tikzcd}$$

From the "naturality of the connecting homomorphism ∂" [123, Chapter 6] and (2.5), we get the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathfrak{m}}(\mathrm{Sym}(E))(-1,-d+\mu) \arrow[r, "\partial"] \arrow[d, "g_2"] & H^2_{\mathfrak{m}}(S)(-2,-d) \arrow[r, "g_1"] \arrow[d, "g_2"] & H^2_{\mathfrak{m}}(S)(-1,-d+\mu) \arrow[d, "g_2"] \\ 0 \arrow[r] & H^1_{\mathfrak{m}}(\mathrm{Sym}(E)) \arrow[r, "\partial"] & H^2_{\mathfrak{m}}(S)(-1,-\mu) \arrow[r, "g_1"] & H^2_{\mathfrak{m}}(S). \end{tikzcd}$$

From this diagram and (2.4), we get the exact sequence

$$0 \to \mathcal{K} \to \operatorname{Ker}\Big(H^2_{\mathfrak{m}}(S)(-2,-d) \xrightarrow{g_2} H^2_{\mathfrak{m}}(S)(-1,-\mu)\Big) \xrightarrow{g_1} \operatorname{Ker}\Big(H^2_{\mathfrak{m}}(S)(-1,-d+\mu) \xrightarrow{g_2} H^2_{\mathfrak{m}}(S)\Big),$$

from which we finally identify

$$\mathcal{K} \cong \Big\{ w \in H^2_{\mathfrak{m}}(S)(-2,-d) \mid g_1 \cdot w = 0 \text{ and } g_2 \cdot w = 0 \Big\}.$$

□

Corollary 2.6. *Adopt Setup 2.1. The following statements hold:*

(i) *For* $p \geqslant 2$ *the graded part* $\mathcal{K}_{p,*}$ *is a finite dimensional* $\Bbbk$*-vector space with* $\mathcal{K}_{p,d-2} \neq 0$ *and* $\mathcal{K}_{p,q} = 0$ *for* $q > d-2$.

(ii) $\mathcal{K}_{*,d-2} \cong U(-2)$ *is an isomorphism of graded* U*-modules.*

Proof. (i) For any $q > d-2$ we have $q-d > -2$, and so $H^2_{\mathfrak{m}}(R)_{q-d} = 0$ which implies $\mathcal{K}_{p,q} = 0$. If $q = d-2$ then $H^2_{\mathfrak{m}}(R)_{q-d} = \Bbbk \cdot \frac{1}{x_1x_2}$, and so it follows that $\mathcal{K}_{p,q} \neq 0$ since $x_1 \cdot \frac{1}{x_1x_2} = 0$ and $x_2 \cdot \frac{1}{x_1x_2} = 0$.

(ii) It follows from the fact that $S \cdot \frac{1}{x_1x_2} \cong \Bbbk[T_1,T_2,T_3] = U$. □

In Corollary 2.6 we have seen that the maximal x-degree of every graded part $\mathcal{K}_{p,*}$ is the same and equal to $d-2$, but for the minimal x-degree of $\mathcal{K}_{p,*}$ there is no such nice characterization. In Section 2.3 under the assumption of working over a field of characteristic zero, we shall relate the minimal x-degree with the integral roots of certain b-functions.

2.2 Translation into D-modules

The core of this section is to translate our problem into D-modules. A good introduction to the theory of D-modules can be found in [13] or [37]. The section is divided into two subsections, a first one containing some notations and definitions regarding D-modules that we shall use for the rest of this chapter, and a second one containing our translation.

Notations

For the rest of this chapter we work over a field $\mathbb{F}$ of characteristic zero, and from now on we shall use the following setup.

Setup 2.7. *Adopt Setup 2.1 and change the arbitrary field* $\Bbbk$ *for a field* $\mathbb{F}$ *of characteristic zero.*

We introduce the ring of $\mathbb{F}$-linear differential operators over $R = \mathbb{F}[x_1,x_2]$, which in our characteristic zero case coincides with the Weyl algebra.

Definition 2.8. *The Weyl algebra* $D = A_2(\mathbb{F})$ *is defined as a quotient of the free algebra* $\mathbb{F}{<}x_1, x_2, \partial_1, \partial_2{>}$ *by the two sided ideal generated by the relations*

$$x_i x_j = x_j x_i, \qquad \partial_i \partial_j = \partial_j \partial_i, \qquad \partial_i x_j = x_j \partial_i + \delta_{ij},$$

where δ_{ij} *is Kronecker's symbol.*

The D-module structure of R is given by: for any $f \in R$, the operator x_i is the usual multiplication $x_i \bullet f = x_i f$ and the operator ∂_i is the differentiation $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$. We shall always stress the action of the Weyl algebra by using the symbol " $\bullet$". Thus, for instance, if we regard $x_1 \in R$ then we have $\partial_1 \bullet x_1 = 1$, but instead for $x_1 \in D$ we have $\partial_1 x_1 = x_1 \partial_1 + 1$.

Of particular interest are the holonomic D-modules. A finitely generated left D-module $M \neq 0$ is said to be holonomic if it has Bernstein dimension $d(M) = 2$, or equivalently, if $\mathrm{Ext}^i_D(M, D)$ vanishes for all $i \neq 2$. A left D-ideal J is said to be holonomic when D/J is holonomic.

All the modules in the Čech complex are localizations of R, thus by defining the D-module structure of any localization R_f of R, the local cohomology modules obtain a natural structure as D-modules (see e.g. [88, Lecture 23]). For any localization R_f the D-module structure is defined by

$$x_i \bullet \frac{g}{f^k} = x_i \frac{g}{f^k} \qquad \text{and} \qquad \partial_i \bullet \frac{g}{f^k} = \frac{1}{f^k}\frac{\partial g}{\partial x_i} - \frac{kg}{f^{k+1}}\frac{\partial f}{\partial x_i}.$$

Due to the non-commutativity of D, we need to take some care with the maps of left or right D-modules. Let $A \in D^{r \times s}$ be an $r \times s$ matrix with entries in D. Multiplying with A gives us a map of left D-modules,

$$D^r \xrightarrow{\cdot A} D^s \quad : \quad [\ell_1, \ldots, \ell_r] \mapsto [\ell_1, \ldots, \ell_r] \cdot A,$$

where we regard D^r and D^s as row vectors.

The matrix $A \in D^{r \times s}$ also defines a map of right D-modules in the opposite direction,

$$(D^s)^T \xrightarrow{A\cdot} (D^r)^T \quad : \quad \left[\ell'_1, \ldots, \ell'_s\right]^T \mapsto A \cdot \left[\ell'_1, \ldots, \ell'_s\right]^T,$$

where the superscript-T means that $(D^s)^T$ and $(D^r)^T$ are considered as column vectors. The right D-module $(D^s)^T$ may be regarded as the dual module $\mathrm{Hom}_D(D^s, D)$. Applying $\mathrm{Hom}_D(-, D)$ to the map $D^r \xrightarrow{\cdot A} D^s$ of left D-modules induces the map $(D^s)^T \xrightarrow{A\cdot} (D^r)^T$ of right D-modules.

We have an equivalence between the category of left D-modules and the category of right D-modules, given by the algebra involution

$$D \xrightarrow{\tau} D \quad : \quad x^\alpha \partial^\beta \mapsto (-\partial)^\beta x^\alpha.$$

The map τ is called the standard transposition. For instance, given a left D-module D^r/M_0 its corresponding standard transposition is the right D-module

$$\tau\Big(\frac{D^r}{M_0}\Big) = \frac{D^r}{\tau(M_0)}, \qquad \tau(M_0) = \{\tau(L) \mid L \in M_0\}.$$

See [117] for more details on the standard transposition τ.

Finally, to describe all the graded parts $\mathcal{K}_{p,*}$ together, we need to define a larger algebra to work in.

Definition 2.9. *We define $\mathcal{T}$ as a polynomial ring in the three variables T_1, T_2, T_3 over the Weyl algebra, that is $\mathcal{T} = A_2(\mathbb{F})[T_1, T_2, T_3] = \mathbb{F}[x_1, x_2]<\partial_1, \partial_2>[T_1, T_2, T_3]$.*

We extend the standard transposition τ over $\mathcal{T}$ by making $\tau(T_i) = T_i$. This algebra $\mathcal{T}$ is naturally a graded U-module with grading on the T_i's, and by $\mathcal{T}_p$ we denote the free D-module spanned by the monomials $\mathbf{T}^\gamma$ with $|\gamma| = p$, that is, $\mathcal{T}_p = D^{\binom{p+2}{2}}$. Also, for technical purposes we need to introduce the subcategory $\mathcal{M}^l_U(\mathcal{T})$ of left $\mathcal{T}$-modules with an underlying structure of graded U-module. The subcategory $\mathcal{M}^r_U(\mathcal{T})$ of U-graded right $\mathcal{T}$-modules can be defined in a completely similar way. We essentially follow the exposition of [19, Section 1.5].

Definition 2.10. *We say that a left $\mathcal{T}$-module M has an underlying structure of graded U-module (or simply that it is U-graded) when it has a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$, where each M_i is a left D-module and $\mathcal{T}_p \bullet M_i \subset M_{i+p}$.*

Definition 2.11. *The category $\mathcal{M}^l_U(\mathcal{T})$, has as objects the left $\mathcal{T}$-modules with an underlying structure of graded U-module. A morphism $\varphi : M \to N$ in $\mathcal{M}^l_U(\mathcal{T})$ is a homomorphism of left $\mathcal{T}$-modules satisfying $\varphi(M_i) \subset N_i$ for all $i \in \mathbb{Z}$.*

If M belongs to $\mathcal{M}^l_U(\mathcal{T})$, then $M(i) \in \mathcal{M}^l_U(\mathcal{T})$ denotes the U-graded left $\mathcal{T}$-module with grading given by $M(i)_n = M_{i+n}$. All the following assertions follow from the fact that the T_i's are central in $\mathcal{T}$.

Since each module $M \in \mathcal{M}^l_U(\mathcal{T})$ is a homomorphic image (in $\mathcal{M}^l_U(\mathcal{T})$) of a free module (in $\mathcal{M}^l_U(\mathcal{T})$) of the form $\bigoplus \mathcal{T}(i)$ (simply by choosing homogeneous generators of M), then the category $\mathcal{M}^l_U(\mathcal{T})$ has enough projectives. Thus, every module $M \in \mathcal{M}^l_U(\mathcal{T})$ has a free resolution in $\mathcal{M}^l_U(\mathcal{T})$, and this fact allows us to define derived functors in $\mathcal{M}^l_U(\mathcal{T})$ (see e.g. [123, Chapter 6]).

Let $M \in \mathcal{M}^r_U(\mathcal{T})$ be a U-graded right $\mathcal{T}$-module and $N \in \mathcal{M}^l_U(\mathcal{T})$ be a U-graded left $\mathcal{T}$-module. Then from the non-commutativity of D follows that the tensor product $M \otimes_\mathcal{T} N$ has only a structure of graded U-module; its homogeneous component $(M \otimes_\mathcal{T} N)_n$ is generated (as an $\mathbb{F}$-vector space) by the elements $u \otimes_\mathcal{T} v$ with $u \in M_i$, $v \in N_j$ and $i + j = n$. Using that each module in $\mathcal{M}^l_U(\mathcal{T})$ or in $\mathcal{M}^r_U(\mathcal{T})$ has a free resolution (in $\mathcal{M}^l_U(\mathcal{T})$ or in $\mathcal{M}^r_U(\mathcal{T})$), then (the $\mathbb{F}$-vector space) $\mathrm{Tor}^\mathcal{T}_i(M, N)$ has a natural structure of graded U-module for any $i \geqslant 0$. We shall use the notation $^*\mathrm{Tor}^\mathcal{T}_i(M, N)$ to stress its graded structure as a U-module.

Let $M, N \in \mathcal{M}^l_U(\mathcal{T})$ be U-graded left $\mathcal{T}$-modules. A homomorphism of left $\mathcal{T}$-modules $\varphi : M \to N$ is called homogeneous of degree i if $\varphi(M_n) \subset N_{n+i}$ for all $n \in \mathbb{Z}$. We denote by

$\mathrm{Hom}_i(M,N)$ the $\mathbb{F}$-vector space of homogeneous homomorphisms of degree i. The $\mathbb{F}$-vector subspaces $\mathrm{Hom}_i(M,N)$ of $\mathrm{Hom}_{\mathcal{T}}(M,N)$ form a direct sum, and we have that

$$^*\mathrm{Hom}_{\mathcal{T}}(M,N) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(M,N)$$

is naturally a graded $\mathcal{U}$-module. Also, when M is finitely generated we have that $^*\mathrm{Hom}_{\mathcal{T}}(M,N) = \mathrm{Hom}_{\mathcal{T}}(M,N)$.

For any $N \in \mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$ we define $^*\mathrm{Ext}^i_{\mathcal{T}}(M,N)$ as the i-th right derived functor of $^*\mathrm{Hom}_{\mathcal{T}}(-,N)$ in $\mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$. Hence, given a projective resolution $P_\bullet$ of M in $\mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$, we have

$$^*\mathrm{Ext}^i_{\mathcal{T}}(M,N) \cong H^i(^*\mathrm{Hom}_{\mathcal{T}}(P_\bullet,N)),$$

for all $i \geqslant 0$. A particular and important case is when $N = \mathcal{T}$, since $\mathcal{T}$ can be seen as a bimodule then we have that $^*\mathrm{Ext}^i_{\mathcal{T}}(M,\mathcal{T})$ is a module in the category $\mathcal{M}^r_{\mathcal{U}}(\mathcal{T})$ of right $\mathcal{T}$-modules with a structure of graded $\mathcal{U}$-module.

The Weyl algebra $D = A_2(\mathbb{F})$ is a left Noetherian ring (see e.g. [13, Proposition 2.8, page 6]), then from the Hilbert basis theorem (see e.g. [123, Theorem 3.21]) we have that $\mathcal{T}$ is also a left Noetherian ring. Thus, for $M \in \mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$ finitely generated we can find a resolution in $\mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$ made-up of finitely generated free modules, and so we have that $^*\mathrm{Ext}^i_{\mathcal{T}}(M,N) = \mathrm{Ext}^i_{\mathcal{T}}(M,N)$. We shall use the notation $^*\mathrm{Ext}^i_{\mathcal{T}}(M,N)$ to emphasize its graded structure as a $\mathcal{U}$-module.

The translation

We can see that $S = \bigoplus_\gamma R\mathbf{T}^\gamma$ and $H^2_m(S) = \bigoplus_\gamma H^2_m(R)\mathbf{T}^\gamma$ both belong to the category $\mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$ of $\mathcal{U}$-graded left $\mathcal{T}$-modules.

Proposition 2.12. *(i) The left $\mathcal{T}$-module $H^2_m(S)$ is cyclic with generator $\frac{1}{x_1x_2}$ and presentation*

$$0 \to \mathcal{T}(x_1,x_2) \to \mathcal{T} \xrightarrow{\bullet\frac{1}{x_1x_2}} H^2_m(S) \to 0.$$

(ii) The left $\mathcal{T}$-module S is cyclic with generator 1 and presentation

$$0 \to \mathcal{T}(\partial_1,\partial_2) \to \mathcal{T} \xrightarrow{\bullet 1} S \to 0.$$

Proof. (i) To prove that $\frac{1}{x_1x_2}$ is a generator of $H^2_m(S)$ it is enough to show that any monomial $\frac{1}{x_1^{\alpha_1}x_2^{\alpha_2}}T_1^{\gamma_1}T_2^{\gamma_2}T_3^{\gamma_3}$ belongs to $\mathcal{T}\bullet\frac{1}{x_1x_2}$, but this is obtained from the fact that $\mathrm{char}(\mathbb{F}) = 0$ and the following identity

$$\partial_1^{\alpha_1-1}\partial_2^{\alpha_2-1}T_1^{\gamma_1}T_2^{\gamma_2}T_3^{\gamma_3}\bullet\frac{1}{x_1x_2} = (-1)^{\alpha_1+\alpha_2}\frac{(\alpha_1-1)!(\alpha_2-1)!}{x_1^{\alpha_1}x_2^{\alpha_2}}T_1^{\gamma_1}T_2^{\gamma_2}T_3^{\gamma_3}.$$

On the other hand, the annihilator of $\frac{1}{x_1x_2}$ is given by the left ideal $\mathcal{T}(x_1, x_2)$.

(ii) Follows in a similar way by taking 1 as the generator. □

From this previous proposition we get the isomorphisms of left $\mathcal{T}$-modules

$$S \cong \mathcal{T}/\mathcal{T}(\partial_1, \partial_2) \qquad \text{and} \qquad H^2_m(S) \cong \mathcal{T}/\mathcal{T}(x_1, x_2).$$

Remark 2.13. *For any* $w \in H^2_m(S)$ *we have that* $g_i \bullet w = g_i \cdot w$*, and so we have that*

$$\{w \in H^2_m(S) \mid g_1 \bullet w = 0 \text{ and } g_2 \bullet w = 0\} = \{w \in H^2_m(S) \mid g_1 \cdot w = 0 \text{ and } g_2 \cdot w = 0\},$$

which gives us that we can enlarge S *into* $\mathcal{T}$ *and still recover the same object* $\mathcal{K}$ *that we are interested in.*

At the moment we have a description of $\mathcal{K}$ as the set of elements in $H^2_m(S)$ annihilated by the polynomials g_1 and g_2, but certainly it would be interesting to have a description as the set of elements in S annihilated by certain differential operators. To achieve this, we use the Fourier transform (see [37, Section 5.2]).

Definition 2.14. *By* $\mathcal{F}$ *we denote the automorphism on* $\mathcal{T}$ *defined by*

$$\mathcal{F}(x_i) = \partial_i, \quad \mathcal{F}(\partial_i) = -x_i, \quad \mathcal{F}(T_i) = T_i.$$

Notation 2.15. *For the rest of this chapter we shall use the notations* $L_1 = \mathcal{F}(g_1)$ *and* $L_2 = \mathcal{F}(g_2)$.

Lemma 2.16. *The* $\mathbb{F}$*-vector space* $\mathrm{Sol}(L_1, L_2; S) = \{h \in S \mid L_1 \bullet h = 0 \text{ and } L_2 \bullet h = 0\}$ *has a structure of* S*-module given by the twisting of the Fourier transform:*

$$\textit{let } f \in S,\ h \in \mathrm{Sol}(L_1, L_2; S) \quad \textit{then we define} \quad f \cdot h = \mathcal{F}(f) \bullet h. \tag{2.7}$$

Also it has a bigraded structure induced from S, that is,

$$\mathrm{Sol}(L_1, L_2; S) = \bigoplus_{i \geqslant 0, j \leqslant 0} \mathrm{Sol}(L_1, L_2; S)_{i,j}, \tag{2.8}$$

where $\mathrm{Sol}(L_1, L_2; S)_{i,j} = \mathrm{Sol}(L_1, L_2; S) \cap S_{i,-j}$.

Proof. For any $h \in \mathrm{Sol}(L_1, L_2; S)$ we have that $T_i h \in \mathrm{Sol}(L_1, L_2; S)$ and $\partial_i \bullet h \in \mathrm{Sol}(L_1, L_2; S)$, therefore it follows that $\mathrm{Sol}(L_1, L_2; S)$ has a structure of S-module given by (2.7).

The bigraded decomposition of (2.8) comes from the fact that L_1 and L_2 are bihomogeneous, both with degree 1 on the T_i's, and degree μ and $d - \mu$ respectively on the ∂_i's. We need to index with non-positive integers $j \leqslant 0$ on the x-degree to satisfy the condition $x_i \cdot \mathrm{Sol}(L_1, L_2; S)_{i,j} \subset \mathrm{Sol}(L_1, L_2; S)_{i,j+1}$. □

Notation 2.17. *We denote* $\mathfrak{S} = \mathrm{Sol}(L_1, L_2; S)_{\mathcal{F}}$ *to stress the bigraded* S*-module structure induced on* $\mathrm{Sol}(L_1, L_2; S)$ *by the twisting of the Fourier transform* $\mathcal{F}$*.*

Theorem 2.18. *Adopt Setup 2.7. We have the following isomorphism of bigraded* S*-modules*

$$\mathcal{K} \cong \mathfrak{S}(-2, -d+2),$$

induced by the Fourier transform $\mathcal{F}$*.*

Proof. We divide the proof into three short steps.

Step 1. We define the following two canonical maps

$$\Pi_x : \mathcal{T} \to \mathcal{T}/\mathcal{T}(\partial_1, \partial_2)\ (\cong S), \qquad \Pi_\partial : \mathcal{T} \to \mathcal{T}/\mathcal{T}(x_1, x_2)\ (\cong H^2_{\mathfrak{m}}(S)).$$

For any $z \in \mathcal{T}$ we have the equivalence

$$\big(\ z\ \in\ \mathcal{T}(x_1, x_2)\ \big) \iff \big(\ \mathcal{F}(z)\ \in\ \mathcal{T}(\partial_1, \partial_2)\ \big),$$

therefore we get an induced isomorphism $\overline{\mathcal{F}} : H^2_{\mathfrak{m}}(S) \to S_{\mathcal{F}}$ of left $\mathcal{T}$-modules, where $S_{\mathcal{F}}$ denotes S twisted by $\mathcal{F}$. This isomorphism satisfies

$$\overline{\mathcal{F}}(\Pi_\partial(z)) = \Pi_x(\mathcal{F}(z)).$$

Step 2. For any $z \in \mathcal{T}$ we have the following equivalences

$$\begin{pmatrix} g_1 \bullet \Pi_\partial(z) = 0 \\ g_2 \bullet \Pi_\partial(z) = 0 \end{pmatrix} \iff \begin{pmatrix} g_1 z\ \in\ \mathcal{T}(x_1, x_2) \\ g_2 z\ \in\ \mathcal{T}(x_1, x_2) \end{pmatrix} \iff \begin{pmatrix} \mathcal{F}(g_1)\mathcal{F}(z)\ \in\ \mathcal{T}(\partial_1, \partial_2) \\ \mathcal{F}(g_2)\mathcal{F}(z)\ \in\ \mathcal{T}(\partial_1, \partial_2) \end{pmatrix} \iff$$
$$\iff \begin{pmatrix} L_1 \bullet \Pi_x(\mathcal{F}(z)) = 0 \\ L_2 \bullet \Pi_x(\mathcal{F}(z)) = 0 \end{pmatrix} \iff \begin{pmatrix} L_1 \bullet \overline{\mathcal{F}}(\Pi_\partial(z)) = 0 \\ L_2 \bullet \overline{\mathcal{F}}(\Pi_\partial(z)) = 0 \end{pmatrix}.$$

Therefore $\overline{\mathcal{F}}$ induces an isomorphism of S-modules

$$\left\{ w \in H^2_{\mathfrak{m}}(S) \mid g_1 \bullet w = 0 \text{ and } g_2 \bullet w = 0 \right\} \cong \mathrm{Sol}(L_1, L_2; S)_{\mathcal{F}}.$$

Step 3. From the definition of $\mathcal{F}$ we have that $\overline{\mathcal{F}}$ is homogeneous of degree 0 on the T_i's. On the other hand, we have that $\overline{\mathcal{F}}$ makes a shift degree of 2 in the x_i's since it sends

$$\frac{1}{x_1^{\alpha_1} x_2^{\alpha_2}} = (-1)^{\alpha_1 + \alpha_2} \frac{\partial_1^{\alpha_1 - 1} \partial_2^{\alpha_2 - 1}}{(\alpha_1 - 1)!(\alpha_2 - 1)!} \bullet \frac{1}{x_1 x_2} \in H^2_{\mathfrak{m}}(R)$$

to

$$\mathcal{F}\left((-1)^{\alpha_1 + \alpha_2} \frac{\partial_1^{\alpha_1 - 1} \partial_2^{\alpha_2 - 1}}{(\alpha_1 - 1)!(\alpha_2 - 1)!} \right) = \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1}}{(\alpha_1 - 1)!(\alpha_2 - 1)!} \in R.$$

Then adding the shift degrees $(0, 2)$ to Theorem 2.5 we obtain the result. □

Notation 2.19. *Since both* L_1 *and* L_2 *are linear on the* T_i*'s, then we get that* $\mathcal{T}(L_1, L_2) \in \mathcal{M}_U^l(\mathcal{T})$ *and* $\mathcal{T}/\mathcal{T}(L_1, L_2) \in \mathcal{M}_U^l(\mathcal{T})$. *We denote this last quotient by* $Q = \mathcal{T}/\mathcal{T}(L_1, L_2)$.

Before finishing this section we present an isomorphism of graded U-modules that will be the starting point of Section 2.4.

Proposition 2.20. *Adopt Setup 2.7. We have the following isomorphism of graded* U*-modules*

$$\mathcal{K} \cong {}^*\mathrm{Hom}_{\mathcal{T}}\big(Q, S\big)(-2).$$

Proof. The following isomorphism of $\mathbb{F}$-vector spaces

$$\mathrm{Hom}_{\mathcal{T}}\big(\mathcal{T}/\mathcal{T}(L_1, L_2), S\big) \cong \big\{h \in S \mid L_1 \bullet h = 0 \text{ and } L_2 \bullet h = 0\big\} = \mathrm{Sol}(L_1, L_2; S)$$

follows in the same way as in [37, Chapter 6, Theorem 1.2]. From the discussions of Section 2.2, we actually have an isomorphism ${}^*\mathrm{Hom}_{\mathcal{T}}(Q, S) \cong \mathrm{Sol}(L_1, L_2; S)$ of graded U-modules. The shifting of degree follows from Theorem 2.18. □

2.3 The bigraded structure of $\mathcal{K}$ and its relation with b-functions

For organizational purposes we have divided this section into two subsections. In the first one, we use the theory of D-modules (specifically, the existence of b-functions) to obtain an upper bound for the degree of the polynomial solutions of the system of differential equations $\mathrm{Sol}(L_1, L_2; S)$, then from Theorem 2.18 follows a lower bound in the possible x-degree. In the second subsection, using the local duality theorem for graded modules we prove that this bound it is always strict.

Polynomial solutions

Our treatment in this subsection follows [117, Section 2], but we need to make some variations since the algorithm given there is restricted to holonomic ideals inside the Weyl algebra. We use [125] as our reference regarding Gröbner deformations and the algorithmic aspects of D-modules.

Setup 2.21. *Adopt Setup 2.7. We fix the integers* $p \geqslant 2$, $m = \binom{p}{2}$ *and* $n = \binom{p+1}{2}$. *The graded part* $\mathfrak{S}_{p-2,*}$ *is given as the solution set of the system of differential equations*

$$V = \big\{h = (h_1, \ldots, h_m) \in R^m \mid [L_1] \bullet h = 0 \textit{ and } [L_2] \bullet h = 0\big\}, \tag{2.9}$$

where $[L_i] \in D^{n \times m}$ *is an* $n \times m$ *matrix with entries in* D *and induced by restricting* L_i *to the monomials* $\mathbf{T}^\gamma$ *of degree* $|\gamma| = p - 2$. *We join both matrices in a single matrix* $H \in D^{2n \times m}$ *defined by*

$$H = \left(\frac{L_1}{L_2}\right), \tag{2.10}$$

then equivalently we can write $V = \{h = (h_1, \ldots, h_m) \in R^m \mid H \bullet h = 0\}$. *We define* $N \subset D^m$ *as the left* D*-module given as the image of* H*, i.e.,* $N = D^{2n} \cdot H$. *With* M *we denote the quotient module* $M = D^m/N$*, we have an isomorphism* $\mathrm{Hom}_D(M, R^m) \cong V$ *of* $\mathbb{F}$*-vector spaces.*

Example 2.22. *We give the explicit form of the system of differential equations (2.9) in the cases* $p = 2$ *and* $p = 3$*. Suppose that* $L_1 = a_1T_1 + a_2T_2 + a_3T_3$ *and* $L_2 = b_1T_1 + b_2T_2 + b_3T_3$*. For* $p = 2$ *we have that* $h = (h_1) \in S_0 = R$*, and the equations* $L_1 \bullet h = 0$ *and* $L_2 \bullet h = 0$ *can be expressed as*

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet (h_1) = 0 \quad \textit{and} \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \bullet (h_1) = 0,$$

and in this case we have that N *is actually the left ideal* $D(a_1, a_2, a_3, b_1, b_2, b_3)$*. When* $p = 3$*, we have that* $h = (h_1, h_2, h_3) \in S_1 = RT_1 + RT_2 + RT_3$*, and sorting the monomials* $\mathbf{T}^\gamma$ *in lexicographical order we get that the equations* $L_1 \bullet h = 0$ *and* $L_2 \bullet h = 0$ *can be expressed as*

$$\begin{array}{c} \\ T_1^2 \\ T_1T_2 \\ T_1T_3 \\ T_2^2 \\ T_2T_3 \\ T_3^2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_1 & 0 \\ a_3 & 0 & a_1 \\ 0 & a_2 & 0 \\ 0 & a_3 & a_2 \\ 0 & 0 & a_3 \end{pmatrix} \end{array} \bullet \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 0 \quad \text{and} \quad \begin{array}{c} \\ T_1^2 \\ T_1T_2 \\ T_1T_3 \\ T_2^2 \\ T_2T_3 \\ T_3^2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ b_3 & 0 & b_1 \\ 0 & b_2 & 0 \\ 0 & b_3 & b_2 \\ 0 & 0 & b_3 \end{pmatrix} \end{array} \bullet \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = 0.$$

Proposition 2.23. *The left* D*-module* $M = D^m/N$ *is holonomic.*

Proof. From the exact sequence (2.16) of Proposition 2.35 we take the graded part p in the T_i's, which gives the following exact sequence of left D-modules

$$0 \to \mathcal{T}_{p-2} \xrightarrow{\cdot A} \mathcal{T}_{p-1}^2 \xrightarrow{\cdot B} \mathcal{T}_p \to Q_p \to 0, \tag{2.11}$$

where $A = \left(-[L_2]^T \mid [L_1]^T\right)$ and $[L_i]^T$ represent the transpose of the matrix $[L_i]$ from Setup 2.21.[1]

Applying the functor $\mathrm{Hom}_D(-, D)$ we obtain the following complex of right D-modules

$$0 \to (\mathcal{T}_p)^T \xrightarrow{B\cdot} (\mathcal{T}_{p-1}^2)^T \xrightarrow{A\cdot} (\mathcal{T}_{p-2})^T \to 0,$$

where the cokernel of the last map $A\cdot$ is $\mathrm{Ext}_D^2(Q_p, D)$, but from the form of A this coincides with the standard transposition $\tau(M)$ of M, that is, $\mathrm{Ext}_D^2(Q_p, D) \cong \tau(M)$. Finally, from [13, Lemma

[1]With the same notations of Example 2.22, when $p = 2$ we have $A = \begin{pmatrix} -b_1 & -b_2 & -b_3 & | & a_1 & a_2 & a_3 \end{pmatrix} \in D^{1\times 6}$.

7.3, page 73] we have that $\mathrm{Ext}^2_D(Q_p, D)$ is a holonomic right D-module, and this clearly implies the holonomicity of $\tau(M)$ and M. □

We recall the notion of Gröbner deformations (see [125, Section 1.1]). For a given weight $w = (w_1, w_2)$ and an element $\ell = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta \in D$, we denote by $\mathrm{in}_{(-w,w)}(\ell)$ the initial form of ℓ with respect to w and it is defined as

$$\mathrm{in}_{(-w,w)}(\ell) = \sum_{-\alpha\cdot w+\beta\cdot w \text{ is maximum}} c_{\alpha,\beta} x^\alpha \partial^\beta. \tag{2.12}$$

We are interested in the weight $w = (-1, -1)$ that makes $\deg(x_i) = 1$ and $\deg(\partial_i) = -1$, thus we drop the subscript $(-w, w)$ and the definition of the initial form (2.12) turns into

$$\mathrm{in}(\ell) = \sum_{|\alpha-\beta| \text{ is maximum}} c_{\alpha,\beta} x^\alpha \partial^\beta. \tag{2.13}$$

Definition 2.24. [125, Corollary 1.1.2, Definition 1.1.3] *Let $J \subset D$ be a left ideal, then the $\mathbb{F}$-vector space*

$$\mathrm{in}(J) = \mathbb{F} \cdot \left\{ \mathrm{in}(\ell) \mid \ell \in J \right\}$$

is a left ideal in D and it is called the initial ideal of J.

Definition 2.25. [125, Definition 5.1.1] *Let $J \subset D$ be a holonomic left ideal. The elimination ideal*

$$\mathrm{in}(J) \cap \mathbb{F}[-x_1\partial_1 - x_2\partial_2] \tag{2.14}$$

is principal in the univariate polynomial ring $\mathbb{F}[s]$, where $s = -x_1\partial_1 - x_2\partial_2$. The generator $b_J(s)$ of the principal ideal (2.14) is called the b-function of J.

An important fact is that the b-function of a holonomic ideal is a non-zero polynomial (see e.g. [125, Theorem 5.1.2]). Now we present a suitable definition for the b-function of a left D-module, which is essentially the same as the one given in [116, Section 4] (see [116, Lemma 4.2]).

Definition 2.26. *Let M' be a holonomic left D-module given as the quotient module $M' = D^r/N'$. For each $i = 1, \dots, r$ with the canonical projection $\pi_i : D^r \to D$ of D^r onto the i-th component e_i, we define a left D-ideal*

$$J_i = \pi_i(N' \cap D \cdot e_i) = \{\ell \in D \mid (0, \dots, \underbrace{\ell}_{i\text{-}th}, \dots, 0) \in N'\}.$$

Then, the b-function of M' is given as the least common multiple of the b-functions of the D-ideals J_i, that is,

$$b_{M'}(s) = \mathrm{LCM}_{i=1,\dots,r}\left(b_{J_i}(s)\right).$$

In this previous definition for each $i = 1, \dots, r$, the canonical injection $D/J_i \hookrightarrow D^r/N'$ implies that each ideal J_i is holonomic, and so we get that the b-function of a holonomic module is a non-zero polynomial. Before proving the main result of this subsection, we recall an easy but important lemma.

Lemma 2.27. *Let* $P(s) \in \mathbb{F}[s]$ *be a polynomial in* $s = -x_1\partial_1 - x_2\partial_2$ *and let* $f \in R$ *be a homogeneous polynomial of degree* $\deg(f) = k$*, then we have* $P(s) \bullet f = P(-k)f$.

Proof. It follows from Euler's formula $(x_1\partial_1 + x_2\partial_2) \bullet f = kf$. □

Theorem 2.28. *Adopt Setup 2.7. Consider the* b*-function* $b_M(s)$ *of the holonomic* D*-module* M *defined in Setup 2.21. For any integer* q*, if* $b_M(-d+2+q) \neq 0$ *then we have that* $\mathcal{K}_{p,q} = 0$.

Proof. Suppose by contradiction that $\mathcal{K}_{p,q} \neq 0$, then from Theorem 2.18 there exists $0 \neq h \in \mathfrak{S}_{p-2,-k}$ where $-k = -d + 2 + q$. Indexing this element as in Setup 2.21 we have a non-zero polynomial vector $h = (h_1, \dots, h_m) \in V$ where each polynomial h_i has degree $\deg(h_i) = k$.

For each each $i = 1, \dots, m$, let $b_{J_i}(s)$ be the b-function corresponding to the left D-ideal $J_i = \pi_i(N \cap D \cdot e_i)$. Then we have $b_{J_i}(s) \cdot e_i \bullet h = 0$, which implies $b_{J_i}(s) \bullet h_i = 0$. Using Lemma 2.27 we get $b_{J_i}(-k)h_i = 0$, but since $b_{J_i}(-k) \neq 0$ then we have $h_i = 0$. Finally, we have obtained the contradiction $h = 0$. □

Corollary 2.29. *Adopt Setup 2.7. Let* q *be the lowest possible* x*-degree for an element in the graded part* $\mathcal{K}_{p,*}$*, that is,* $\mathcal{K}_{p,q} \neq 0$ *and* $\mathcal{K}_{p,q-1} = 0$*. Then the polynomial* $s(s+1)\cdots(s+d-2-q)$ *divides the* b*-function* $b_M(s)$.

Proof. Follows from the contrapositive of the previous theorem. □

The equality

In this subsection we shall prove that the approximation given above is actually strict.

Lemma 2.30. *For any* $k \geqslant 0$ *we have the identity*

$$s(s+1)\cdots(s+k) = (-1)^{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} x_1^j x_2^{k+1-j} \partial_1^j \partial_2^{k+1-j}.$$

Thus, we have that

(i) $s(s+1)\cdots(s+k) \in D(\partial_1, \partial_2)^{k+1}$*, where* $D(\partial_1, \partial_2)^{k+1}$ *denotes the left* D*-ideal generated by the elements* $\{\partial_1^{\beta_1}\partial_2^{\beta_2} \mid \beta_1 + \beta_2 = k+1\}$*;*

(ii) $s(s+1)\cdots(s+k)$ *is homogeneous, that is*

$$\mathrm{in}\big(s(s+1)\cdots(s+k)\big) = s(s+1)\cdots(s+k).$$

Proof. We proceed by induction on k. For $k = 0$ it is clear since $s = -x_1\partial_1 - x_2\partial_2$.

First we prove the identity $x_i^{\beta_i}\partial_i^{\beta_i}(x_i\partial_i - \beta_i) = x_i^{\beta_i+1}\partial_i^{\beta_i+1}$ using induction on β_i. For $\beta_i = 0$ it is vacuous, thus we assume that $\beta_i > 0$ and that the statement holds for any non-negative integer smaller than β_i. Hence, we have the equalities

$$\begin{aligned} x_i^{\beta_i}\partial_i^{\beta_i}(x_i\partial_i - \beta_i) &= x_i^{\beta_i}\partial_i^{\beta_i-1}(\partial_i x_i\partial_i - \beta_i\partial_i) \\ &= x_i x_i^{\beta_i-1}\partial_i^{\beta_i-1}(x_i\partial_i - (\beta_i - 1))\partial_i = x_i x_i^{\beta_i}\partial_i^{\beta_i}\partial_i = x_i^{\beta_i+1}\partial_i^{\beta_i+1}. \end{aligned}$$

Then, we can obtain that

$$x_1^j x_2^{k+1-j}\partial_1^j\partial_2^{k+1-j}(s + k + 1) = (-1)\left(x_1^{j+1}x_2^{k+1-j}\partial_1^{j+1}\partial_2^{k+1-j} + x_1^j x_2^{k+2-j}\partial_1^j\partial_2^{k+2-j}\right),$$

and using $\binom{k+1}{j} + \binom{k+1}{j-1} = \binom{k+2}{j}$ the proof of the lemma follows similarly to the usual binomial formula. $\square$

From Setup 2.21 we can define the matrix $F = \mathcal{F}(H) = (\mathcal{F}(H_{i,j})) \in R^{2n\times m}$ (where $m = \binom{p}{2}, n = \binom{p+1}{2}$), that is, the $2n \times m$ matrix with entries in R obtained after applying the Fourier transform to each entry of the matrix H. In a similar way to Setup 2.21, we define the graded R-module $L = R^m/(R^{2n} \cdot F)$ (all the rows of F are homogeneous of degree μ or degree $d - \mu$).

Since $\{g_1, g_2\}$ is a regular sequence in S (see the proof of Proposition 2.35), by restricting the Koszul complex $K_\bullet(g_1, g_2)$ to the graded part p, the module $\mathrm{Sym}_p(I)$ gets the graded free resolution

$$0 \to R(-d)^{\binom{p}{2}} \to R(-\mu)^{\binom{p+1}{2}} \oplus R(-d+\mu)^{\binom{p+1}{2}} \to R^{\binom{p+2}{2}} \to \mathrm{Sym}_p(I) \to 0.$$

Similarly to Proposition 2.23, when we apply $\mathrm{Hom}_R(-, R)$ we get a complex

$$0 \to R^{\binom{p+2}{2}} \to R(\mu)^{\binom{p+1}{2}} \oplus R(d-\mu)^{\binom{p+1}{2}} \to R(d)^{\binom{p}{2}} \to 0, \tag{2.15}$$

where the cokernel of the map on the right is the graded R-module ${}^*\mathrm{Ext}_R^2(\mathrm{Sym}_p(I), R)$. Making a shift degree of $-d$ on the modules of (2.15) gives us a complex that has the module L as the cokernel of the map on the right. Therefore, we have an isomorphism $L(d) \cong {}^*\mathrm{Ext}_R^2(\mathrm{Sym}_p(I), R)$ of graded R-modules.

Now, as an application of the local duality theorem in the graded case (see e.g. [17, Section 14.4] or [19, Section 3.6]) we can prove our sought equality.

Theorem 2.31. *Adopt Setup 2.7. Let $b_M(s)$ be the b-function of the holonomic module M defined in Setup 2.21 and let q be the lowest possible x-degree for an element in the graded part $\mathcal{K}_{p,*}$. Then*

$$b_M(s) = s(s+1)\cdots(s+d-2-q).$$

Proof. From Corollary 2.29 we already know that $s(s+1)\cdots(s+d-2-q) \mid b_M(s)$, then is enough for us to prove that for each $i = 1, \dots, m$ we have

$$s(s+1)\cdots(s+d-2-q) \in \mathrm{in}(J_i) \cap \mathbb{F}[s],$$

where $J_i = \pi_i(N \cap D \cdot e_i)$.

Let $a = \mathrm{end}(L) = \max\{k \mid L_k \neq 0\}$ (since L is a finite length module), then for any $x_1^{\alpha_1} x_2^{\alpha_2}$ with $\alpha_1 + \alpha_2 = a + 1$ we have that

$$x_1^{\alpha_1} x_2^{\alpha_2} e_i = (0, \dots, \underbrace{x_1^{\alpha_1} x_2^{\alpha_2}}_{i\text{-th}}, \dots, 0) \in R^{2n} \cdot F,$$

where $i = 1, \dots, m$ and by an abuse of notation e_i also represents the i-th component of the free R-module R^m. Applying the Fourier transform and using Lemma 2.30 we obtain that

$$s(s+1)\cdots(s+a) \in \mathrm{in}(J_i) \cap \mathbb{F}[s]$$

for each $i = 1, \dots, m$. From the local duality theorem in the graded case, we get the following isomorphisms of graded R-modules

$$\mathcal{K}_{p,*} = H^0_m(\mathrm{Sym}_p(I)) \cong {}^*\mathrm{Hom}_{\mathbb{F}}\big({}^*\mathrm{Ext}^2_R(\mathrm{Sym}_p(I), R(-2)), \mathbb{F}\big) \cong {}^*\mathrm{Hom}_{\mathbb{F}}\big(L(d-2), \mathbb{F}\big).$$

Since the grading of ${}^*\mathrm{Hom}_{\mathbb{F}}\big(L(d-2), \mathbb{F}\big)$ is given by

$${}^*\mathrm{Hom}_{\mathbb{F}}\big(L(d-2), \mathbb{F}\big)_i = {}^*\mathrm{Hom}_{\mathbb{F}}\big(L(d-2)_{-i}, \mathbb{F}\big),$$

we have that $a = d - 2 - q$, and so the statement of theorem follows. □

2.4 Computing Hom with duality

The aim of this section is to compute ${}^*\mathrm{Hom}_{\mathcal{T}}\big(Q, S\big)$ (where $Q = \mathcal{T}/\mathcal{T}(L_1, L_2)$) by means of some duality that was previously used in [140]. In the general Weyl algebra $A_n(\mathbb{F})$, for two holonomic left $A_n(\mathbb{F})$-modules M and N we have the following duality (see e.g. [13, Proposition 4.14, page 58] or [140, Theorem 2.1])

$$\mathrm{Ext}^i_{A_n(\mathbb{F})}(M, N) \cong \mathrm{Tor}^{A_n(\mathbb{F})}_{n-i}\big(\mathrm{Ext}^n_{A_n(\mathbb{F})}(M, A_n(\mathbb{F})),\ N\big),$$

which is one of the main tools used in [140]. Unfortunately we want to work over our previously defined algebra $\mathcal{T}$ and for this we will have to make some variations. Nevertheless, we can achieve the following duality in our case.

Theorem 2.32. *For any i we have the following isomorphism of graded U-modules (see Section 2.2)*

$${}^*\mathrm{Ext}^i_{\mathcal{T}}(Q, S) \cong {}^*\mathrm{Tor}^{\mathcal{T}}_{2-i}\big({}^*\mathrm{Ext}^2_{\mathcal{T}}(Q, \mathcal{T}),\ S\big).$$

To prove this duality we use [13, Chapter 2] as our main source. We start by defining a Bernstein filtration on $\mathcal{T}$ and exploiting the induced graded ring.

Definition-Proposition 2.33. *For any* $i \geqslant 0$ *we define the* $\mathbb{F}$*-vector space* F_i *which is generated by the set of monomials* $\{x_1^{\alpha_1}x_2^{\alpha_2}\partial_1^{\beta_1}\partial_2^{\beta_2}T_1^{\gamma_1}T_2^{\gamma_2}T_3^{\gamma_3} \mid |\alpha|+|\beta|+|\gamma| \leqslant i\}$*, and we denote* $F_{-1}=0$*. Since we have*

(1) $0=F_{-1}\subset F_0\subset F_1\subset F_2\subset\cdots\subset\mathcal{T}$,

(2) $\mathcal{T}=\bigcup_{i\geqslant 0}F_i$,

(3) $F_i\cdot F_j\subset F_{i+j}$,

then $F=\{F_i\}$ *is a filtration of* $\mathcal{T}$*. With this filtration we induce the associated graded ring* $\mathrm{gr}(\mathcal{T})=\bigoplus_{i\geqslant 0}F_i/F_{i-1}$*, which is isomorphic to a polynomial ring of 7 variables with coefficients in* $\mathbb{F}$*. We use the notation*

$$\mathrm{gr}(\mathcal{T})\cong T:=\mathbb{F}[x_1,x_2,\delta_1,\delta_2,T_1,T_2,T_3],$$

where we get a canonical map $\sigma:\mathcal{T}\to T$ *given by* $\sigma(x_i)=x_i$*,* $\sigma(\partial_i)=\delta_i$ *and* $\sigma(T_i)=T_i$*.*

Proof. See [13, Proposition 2.2, page 4] or [37, Theorem 3.1, page 58]. □

We denote by $q_1=\sigma(L_1)$ and $q_2=\sigma(L_2)$ the elements in T corresponding to L_1 and L_2. Here we have, that q_1 and q_2 are bihomogeneous polynomials which are linear on the T_i's, and have degree μ and $d-\mu$ on the δ_i's respectively. But from the graded structure of T, we only see them as homogeneous polynomials having degree $\mu+1$ and $d-\mu+1$ respectively.

A filtration on a left $\mathcal{T}$-module M consists of an increasing sequence of finite dimensional $\mathbb{F}$-subspaces $0=\Gamma_{-1}\subset\Gamma_0\subset\Gamma_1\subset\Gamma_2\subset\cdots$ satisfying $\bigcup\Gamma_i=M$ and the inclusions $F_i\cdot\Gamma_j\subset\Gamma_{i+j}$ for all i and j. With a filtration we get the associated graded $T(=\mathrm{gr}(\mathcal{T}))$-module $\mathrm{gr}_\Gamma(M)=\bigoplus_{i\geqslant 0}\Gamma_i/\Gamma_{i-1}$. We say that $\Gamma=\{\Gamma_i\}$ is a good filtration if $\mathrm{gr}_\Gamma(M)$ is a finitely generated T-module. Using a good filtration we can define a Hilbert-Samuel function, and so we can get a notion of dimension for left $\mathcal{T}$-modules.

Definition-Proposition 2.34. *Given a good filtration* $\Gamma=\{\Gamma_i\}$ *for a finitely generated left* $\mathcal{T}$*-module* M*, there exists a polynomial* $\chi_M^\Gamma(t)=a_dt^d+\cdots+a_1t+a_0$ *with rational coefficients such that* $\dim_{\mathbb{F}}(\Gamma_t)=\chi_M^\Gamma(t)$ *when* $t\gg 0$*. The integer* d *is independent of the good filtration chosen, and we define* $d(M)=d$ *as the Bernstein dimension of* M*.*

Proof. See [13, Section 1.3] or [37, Chapter 9]. □

Since the Hilbert-Samuel function of T is given by $\binom{t+7}{7}$, thus we have $d(\mathcal{T})=7$. Now we want to study the left $\mathcal{T}$-module $Q=\mathcal{T}/\mathcal{T}(L_1,L_2)\in\mathcal{M}_U^l(\mathcal{T})$, and we begin by proving that the Koszul complex gives a free resolution for it in $\mathcal{M}_U^l(\mathcal{T})$.

Proposition 2.35. *The following statements hold.*

(i) The dimension of Q *is* $d(Q) = 5$.

(ii) The following Koszul complex in $\mathcal{M}^l_{\mathcal{U}}(\mathcal{T})$ *is exact*

$$\mathcal{A}_\bullet : \quad 0 \to \mathcal{T}(-2) \xrightarrow{\cdot[-L_2, L_1]} \mathcal{T}(-1)^2 \xrightarrow{\cdot\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]} \mathcal{T} \to Q \to 0. \tag{2.16}$$

Proof. (i) The module Q being a quotient of $\mathcal{T}$ gets a natural good filtration given by $F_i/(F_i \cap \mathcal{T}(L_1, L_2))$ and then we get $\mathrm{gr}(Q) = \bigoplus_{i \geqslant 0} F_i/(F_{i-1} + F_i \cap \mathcal{T}(L_1, L_2)) \cong T/(q_1, q_2)$.

Let $B = \mathbb{F}[x_1, x_2, \delta_1, \delta_2]$, and h_i be the polynomials in B obtained from f_i by making the substitution $x_i \mapsto \delta_i$, i.e., $h_i = \sigma(\mathcal{F}(f_i))$. By the Hilbert-Burch Theorem we have that $J = (h_1, h_2, h_3) \subset B$ is a perfect ideal of height two, and making the substitution $x_i \mapsto \delta_i$ in the resolution (2.2) of I we get a resolution

$$0 \to B^2 \xrightarrow{\Phi} B^3 \xrightarrow{[h_1, h_2, h_3]} J \to 0,$$

where $[q_1, q_2] = [T_1, T_2, T_3] \cdot \Phi$. Hence $T/(q_1, q_2) = \mathrm{Sym}(J)$, and from [133, Corollary 2.2] the Krull dimension is given by $\dim(T/(q_1, q_2)) = \dim(\mathrm{Sym}(J)) = \dim(B) + \mathrm{rank}(J) = 4 + 1 = 5$. Finally, this coincides with the degree of the Hilbert-Samuel polynomial, i.e., $d(Q) = \dim(T/(q_1, q_2)) = 5$ (see e.g. [110, Theorem 13.4]).

(ii) The shifting of degrees in (2.16) are clear since L_1 and L_2 are both linear on the T_i's, then it will be enough to prove exactness of (2.16) just in the category $\mathcal{T}$ (i.e., forgetting the graded structures induced in Section 2.2). So, inside this proof, to avoid confusions the only additional structure that we assume on $\mathcal{T}$ is the Bernstein filtration and the induced graded ring T.

Since $\mathcal{T}$ is non-commutative we should check that (2.16) is even a complex, but fortunately L_1 and L_2 are only defined in the ∂_i and T_i variables, and so $L_1 L_2 - L_2 L_1 = 0$.

The complex (2.16) induces the following graded Koszul complex in T

$$0 \to T(-d-2) \xrightarrow{[-q_2, q_1]} T(-\mu-1) \oplus T(-d+\mu-1) \xrightarrow{\left[\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right]} T \to T/(q_1, q_2) \to 0.$$

Using that $\dim(T/(q_1, q_2)) = 5$ we get that (q_1, q_2) is a T-regular sequence (see e.g. [110, Theorem 17.4]) and so this new complex is exact. Finally, [13, Lemma 3.13, page 46] implies that (2.16) is exact. □

Corollary 2.36. *For any* $j \neq 2$ *we have* ${}^*\mathrm{Ext}^j_{\mathcal{T}}(Q, \mathcal{T}) = 0$*, and* ${}^*\mathrm{Ext}^2_{\mathcal{T}}(Q, \mathcal{T}) \neq 0$.

Proof. Since (2.16) is a free resolution of Q we clearly have ${}^*\mathrm{Ext}^j_{\mathcal{T}}(Q, \mathcal{T}) = 0$ for $j > 2$. On the other hand from [13, Theorem 7.1, page 73] we have that $j(Q) + d(Q) = 7$, where $j(Q) = \inf\{k \mid {}^*\mathrm{Ext}^k_{\mathcal{T}}(Q, \mathcal{T}) \neq 0\}$. Since $d(Q) = 5$, then $j(Q) = 2$ and the statement of the corollary follows. □

Now we are ready to prove the duality that we have claimed at the beginning of this section.

Proof of Theorem 2.32. A resolution of S in $\mathcal{M}^{l}_{\mathrm{U}}(\mathcal{T})$ is given by the Koszul complex

$$\mathcal{B}_\bullet : \quad 0 \to \mathcal{T} \xrightarrow{\cdot[-\partial_2,\partial_1]} \mathcal{T}^2 \xrightarrow{\cdot\left[\begin{smallmatrix}\partial_1\\ \partial_2\end{smallmatrix}\right]} \mathcal{T} \to \mathrm{S} \to 0. \tag{2.17}$$

We define the following third quadrant double complex ${}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_\bullet, \mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_\bullet$:

$$\begin{tikzcd}
{}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_2,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_2 \arrow[d] & {}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_2 \arrow[l] \arrow[d] & {}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_0,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_2 \arrow[l] \arrow[d] \\
{}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_2,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_1 \arrow[d] & {}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_1 \arrow[l] \arrow[d] & {}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_0,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_1 \arrow[l] \arrow[d] \\
{}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_2,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_0 & {}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_0 \arrow[l] & {}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_0,\mathcal{T}) \otimes_{\mathcal{T}} \mathcal{B}_0. \arrow[l]
\end{tikzcd}$$

Thanks to our construction of $\mathcal{M}^{l}_{\mathrm{U}}(\mathcal{T})$ and $\mathcal{M}^{r}_{\mathrm{U}}(\mathcal{T})$, we have that this double complex fits naturally in the category of graded U-modules, that is, all its elements are graded U-modules and all its maps are homogeneous homomorphisms of graded U-modules.

Since each ${}^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_j, \mathcal{T}) \in \mathcal{M}^{r}_{\mathrm{U}}(\mathcal{T})$ is a free module then by computing homology on each column we get that the only row that does not vanish is the last one. On the other hand, from Corollary 2.36 we have that when we compute homology on each row the only column that does not vanish is the leftmost one.

Therefore the spectral sequence determined by the first filtration is given by

$${}^{\mathrm{I}}\mathrm{E}_2^{p,q} = \begin{cases} {}^*\mathrm{Ext}^{p}_{\mathcal{T}}\big(\mathrm{Q}, \mathrm{S}\big) & \text{if } q = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and the spectral sequence determined by the second filtration is given by

$${}^{\mathrm{II}}\mathrm{E}_2^{p,q} = \begin{cases} {}^*\mathrm{Tor}^{\mathcal{T}}_{2-p}\big({}^*\mathrm{Ext}^{2}_{\mathcal{T}}\big(\mathrm{Q}, \mathcal{T}\big), \mathrm{S}\big) & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

From the fact that both spectral sequences collapse we get the following isomorphisms of graded U-modules

$${}^{\mathrm{I}}\mathrm{E}_2^{i,2} \cong \mathrm{H}^{i+2}\big(\mathrm{Tot}\big({}^*\mathrm{Hom}_{\mathcal{T}}\big(\mathcal{A}_\bullet, \mathcal{T}\big) \otimes_{\mathcal{T}} \mathcal{B}_\bullet\big)\big) \cong {}^{\mathrm{II}}\mathrm{E}_2^{i,2},$$

and so we obtain the duality of the theorem. □

Theorem 2.37. *Adopt Setup 2.7. Then, we have the following isomorphism of graded* U*-modules*

$$\mathcal{K} \cong \mathrm{H}^0_{\mathrm{dR}}(\mathrm{Q}) = \{w \in \mathrm{Q} \mid \partial_1 \bullet w = 0 \text{ and } \partial_2 \bullet w = 0\}.$$

In particular, for any integer p *we have an isomorphism of* $\mathbb{F}$*-vector spaces*

$$\mathcal{K}_{p,*} \cong H^0_{dR}(Q_p) = \{w \in Q_p \mid \partial_1 \bullet w = 0 \text{ and } \partial_2 \bullet w = 0\}.$$

Proof. From the resolution (2.16) of Q we get the following complex in $\mathcal{M}^r_U(\mathcal{T})$

$$^*\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}_\bullet, \mathcal{T}) : \quad 0 \to \mathcal{T} \xrightarrow{\left[\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix}\right]\cdot} \mathcal{T}(1)^2 \xrightarrow{[-L_2, L_1]\cdot} \mathcal{T}(2) \to 0.$$

Then, computing the second cohomology of this complex gives that $^*\mathrm{Ext}^2_{\mathcal{T}}(Q, \mathcal{T}) \cong (\mathcal{T}/(L_1, L_2)\mathcal{T})\,(2)$, where $\mathcal{T}/(L_1, L_2)\mathcal{T} = \tau(Q)$ is the standard transposition of Q.

Since the Koszul complex (2.17) gives a resolution of S, then computing the second homology of the Koszul complex $\tau(Q)(2) \otimes_{\mathcal{T}} \mathcal{B}_\bullet$ gives the following isomorphisms of graded U-modules

$$\begin{aligned} ^*\mathrm{Tor}^{\mathcal{T}}_2(^*\mathrm{Ext}^2_{\mathcal{T}}(Q, \mathcal{T}), S) &\cong H_2\big(\tau(Q)(2) \otimes_{\mathcal{T}} \mathcal{B}_\bullet\big) \\ &\cong \{w \in \tau(Q)(2) \mid w \bullet \partial_1 = 0 \text{ and } w \bullet \partial_2 = 0\}. \end{aligned}$$

From the fact that $\tau(T_i) = T_i$, we have an isomorphism of graded U-modules

$$\{w \in \tau(Q)(2) \mid w \bullet \partial_1 = 0 \text{ and } w \bullet \partial_2 = 0\} \cong \{w \in Q(2) \mid \partial_1 \bullet w = 0 \text{ and } \partial_2 \bullet w = 0\},$$

then from Proposition 2.20 and Theorem 2.32 we get the following isomorphisms of graded U-modules

$$\begin{aligned} \mathcal{K} &\cong {}^*\mathrm{Hom}_{\mathcal{T}}(Q, S)(-2) \\ &\cong {}^*\mathrm{Tor}^{\mathcal{T}}_2(^*\mathrm{Ext}^2_{\mathcal{T}}(Q, \mathcal{T}), S)(-2) \\ &\cong \{w \in Q \mid \partial_1 \bullet w = 0 \text{ and } \partial_2 \bullet w = 0\}, \end{aligned}$$

that imply the statement of the theorem. □

2.5 Examples and computations

In this short section we show a simple script in *Macaulay2* [60] that we have implemented to compute the b-function of each D-module M from Setup 2.21. Actually, we have to say that an enormous number of examples and computations led us to believe the equality of Theorem 2.31 in the first place.

```
needsPackage "Dmodules"
bFunctionRees = (I, p) -> (
    R := ring I;
    W := makeWeylAlgebra R;
    T := W[T1, T2, T3], U := QQ[Z1, Z2, Z3];
    A := Fourier (map(W, R, {(vars W)_(0,0),(vars W)_(0,1)})) (res I).dd_2;
```

```
    L := matrix{{T1, T2, T3}} * A;
    L1 := L_(0, 0), L2 := L_(0, 1);
    src := flatten entries (map(T, U, {T1, T2, T3})) basis(p - 2, U);
    dest := flatten entries (map(T, U, {T1, T2, T3})) basis(p - 1, U);
    m := #src, n := #dest;
    H := mutableMatrix(W, m, 2 * n);
    for i from 0 to m - 1 do (
      mult1 := src#i * L1;
      mult2 := src#i * L2;
      for j from 0 to n - 1 do (
        R1 := mult1 // gens ideal(dest#j);
        R2 := mult2 // gens ideal(dest#j);
        H_(i, j) = (map(W, T, {1, 1, 1})) R1_(0, 0);
        H_(i, j + n) = (map(W, T, {1, 1, 1})) R2_(0, 0);
      );
    );
    bM := bFunction(coker matrix H, {-1,-1}, toList(m:0));
    use R;
    bM
)
```

We will carry out a couple of examples to show how we can use Theorem 2.31 to deduce the bigraded structure of $\mathcal{K}$. We can save the previous code in a file that we will call “bFunctionRees.m2”

Example 2.38. *Let* $I = (x^5, x^2y^3, y^5) \subset \mathbb{Q}[x, y]$, *then from* [36] *we know that a minimal set of generators of* $\mathcal{I}$ *is given by*

$$\{y^2T_2 - x^2T_3, \quad y^3T_1 - x^3T_2, \quad xT_2^2 - yT_1T_3, \quad yT_2^3 - xT_1T_3^2, \quad T_2^5 - T_1^2T_3^3\},$$

so a minimal set of generators for $\mathcal{K}$ *is given by*

$$\{xT_2^2 - yT_1T_3, \quad yT_2^3 - xT_1T_3^2, \quad T_2^5 - T_1^2T_3^3\},$$

We make the following session in Macaulay2:

```
i1 :  R = QQ[x,y]
o1 =  R
o1 :  PolynomialRing
i2 :  load "bFunctionRees.m2"
i3 :  I = ideal(x^5, x^2*y^3, y^5)
              5   2 3   5
o3 =  ideal(x , x y , y )
o3 :  Ideal of R
i4 :  for p from 2 to 5 do << factorBFunction bFunctionRees(I, p) << endl;
(s)(s + 1)(s + 2)
(s)(s + 1)(s + 2)
(s)(s + 1)(s + 2)
(s)(s + 1)(s + 2)(s + 3)
```

From Theorem 2.31 we see that for $p = 2, \ldots, 4$ *we have* $\mathcal{K}_{p,q} \neq 0$ *if and only if* $1 \leqslant q \leqslant 3$, *and that* $\mathcal{K}_{5,q} \neq 0$ *if and only if* $0 \leqslant q \leqslant 3$.

Example 2.39. *We assume that in Setup 2.7 we have $\mu = 1$. In this case it is known (see e.g.* [39, Theorem 2.3] *or* [20, Proposition 3.1]*) that the minimal generators of $\mathcal{I}$ have bidegrees*

$$(1,1),\ (1,d-1),\ (2,d-2),\ (3,d-3),\ \ldots,\ (d,0).$$

We can make an interesting session with ideals of this form created randomly, we take the particular case $\mu = 1$ and $d = 7$:

```
i1 :  R = QQ[x,y]
o1 =  R
o1 :  PolynomialRing
i2 :  load "bFunctionRees.m2"
i3 :  A = matrix{{random(1,R),random(6,R)},{random(1,R),random(6,R)},
        {random(1,R),random(6,R)}};
              3      2
o3 :  Matrix R <--- R
i4 :  I =  minors(2, A);
o4 :  Ideal of R
i5 :  assert(codim I == 2);
i6 :  for p from 2 to 7 do << factorBFunction bFunctionRees(I, p) << endl;
(s)
(s)(s + 1)
(s)(s + 1)(s + 2)
(s)(s + 1)(s + 2)(s + 3)
(s)(s + 1)(s + 2)(s + 3)(s + 4)
(s)(s + 1)(s + 2)(s + 3)(s + 4)(s + 5)
```

Here we need to check (`assert(codim I == 2);`*) that the created ideal I has height 2, although it is extremely improbable that this is not the case.*

Part II

Geometry

Chapter 3

Degree and birationality of multi-graded rational maps

In this chapter, we give formulas and effective sharp bounds for the degree of multi-graded rational maps and provide some effective and computable criteria for birationality in terms of their algebraic and geometric properties. We also extend the Jacobian dual criterion to the multi-graded setting. Our approach is based on the study of blow-up algebras, including syzygies, of the ideal generated by the defining polynomials of the rational map. A key ingredient is a new algebra that we call the *saturated special fiber ring*, which turns out to be a fundamental tool to analyze the degree of a rational map.

Note. The results of this chapter are based on joint work with Laurent Busé and Carlos D'Andrea.

3.1 The degree of a multi-graded rational map

In this section we focus on the degree of a rational map between an integral multi-projective variety and an integral projective variety. Our main tool is the introduction of a new algebra which is a saturated version of the special fiber ring. The study of this algebra yields an answer for the degree of a rational map.

Our main result here is Theorem 3.4 where we show that the saturated special fiber ring (Definition 3.3) carries very important information of a rational map. Another fundamental result is Corollary 3.12, which is the main tool for making specific computations.

Preliminaries on multi-graded rational maps

Let $\mathbb{k}$ be a field, $X_1 \subset \mathbb{P}_{\mathbb{k}}^{r_1}, X_2 \subset \mathbb{P}_{\mathbb{k}}^{r_2}, \ldots, X_m \subset \mathbb{P}_{\mathbb{k}}^{r_m}$ and $Y \subset \mathbb{P}_{\mathbb{k}}^{s}$ be integral projective varieties over $\mathbb{k}$. For $i = 1, \ldots, m$, the homogeneous coordinate ring X_i is denoted by $A_i = \mathbb{k}[\mathbf{x_i}]/\mathfrak{a}_i = \mathbb{k}[x_{i,0}, x_{i,1}, \ldots, x_{i,r_i}]/\mathfrak{a}_i$, and $S = \mathbb{k}[y_0, y_1, \ldots, y_s]/\mathfrak{b}$ stands for the homogeneous coordinate ring of Y. We set $R = A_1 \otimes_{\Bbbk} A_2 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_m \cong \Bbbk[\mathbf{x}]/(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$. We also assume that the fiber product $X = X_1 \times_{\Bbbk} X_2 \times_{\Bbbk} \cdots \times_{\Bbbk} X_m$ is an integral scheme (a condition that is satisfied, for instance, when $\Bbbk$ is algebraically closed; see e.g. [57, Lemma 4.23]).

Since we will always work over the field $\Bbbk$, for any two $\Bbbk$-schemes W_1 and W_2 their fiber product $W_1 \times_{\Bbbk} W_2$ will simply be denoted by $W_1 \times W_2$. Similarly, for any $k \geqslant 1$, $\mathbb{P}^k$ will denote the k-th dimensional projective space $\mathbb{P}^k_{\Bbbk}$ over $\Bbbk$.

Let $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y \subset \mathbb{P}^s$ be a rational map defined by $s+1$ multi-homogeneous elements $\mathbf{f} = \{f_0, f_1, \ldots, f_s\} \subset R$ of the same multi-degree. We say that $\mathcal{F}$ is birational if it is dominant and it has an inverse rational map given by a tuple of rational maps

$$\mathcal{G} : Y \dashrightarrow (X_1, X_2, \ldots, X_m).$$

For $i = 1, \ldots, m$, each rational map $Y \dashrightarrow X_i \subset \mathbb{P}^{r_i}$ is defined by $r_i + 1$ homogeneous forms $\mathbf{g}_i = \{g_{i,0}, g_{i,1}, \ldots, g_{i,r_i}\} \subset S$ of the same degree.

Definition 3.1. *The degree of a dominant rational map* $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y$ *is defined as* $\deg(\mathcal{F}) = [K(X) : K(Y)]$*, where* $K(X)$ *and* $K(Y)$ *represent the fields of rational functions of* X *and* Y*, respectively.*

Now, we recall some basic facts about multi-graded rings; we refer the reader to [87], [59] and [58] for more details.

The ring $R = A_1 \otimes_{\Bbbk} A_2 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_m$ has a natural multi-grading given by

$$R = \bigoplus_{(j_1,\ldots,j_m)\in\mathbb{Z}^m} (A_1)_{j_1} \otimes_{\Bbbk} (A_2)_{j_2} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} (A_m)_{j_m}.$$

Let $\mathfrak{N}$ be the multi-homogeneous irrelevant ideal of R, that is

$$\mathfrak{N} = \bigoplus_{j_1>0,\ldots,j_m>0} R_{j_1,\ldots,j_m}.$$

Similarly to the single-graded case, we can define a multi-projective scheme from R. The multi-projective scheme $\mathrm{MultiProj}(R)$ is given by the set of all multi-homogeneous prime ideals in R which do not contain $\mathfrak{N}$, and its scheme structure is obtained by using multi-homogeneous localizations.

For any vector $\mathbf{c} = (c_1, \ldots, c_m)$ of positive integers we can define the multi-Veronese subring

$$R^{(\mathbf{c})} = \bigoplus_{j=0}^{\infty} R_{j\cdot\mathbf{c}},$$

which we see as a standard graded $\Bbbk$-algebra. The canonical injection $R^{(\mathbf{c})} \hookrightarrow R$ induces an isomorphism of schemes $\mathrm{MultiProj}(R) \xrightarrow{\cong} \mathrm{Proj}\left(R^{(\mathbf{c})}\right)$ (see e.g. [66, Exercise II.5.11, Exercise II.5.13]). In particular, if we set $\Delta = (1, \ldots, 1)$, then $\mathrm{Proj}(R^{(\Delta)})$ corresponds with the homogeneous coordinate ring of the image of X under the Segre embedding

$$\mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \times \cdots \times \mathbb{P}^{r_m} \longrightarrow \mathbb{P}^N,$$

with $N = (r_1+1)(r_2+1)\cdots(r_m+1)-1$. Therefore, for any positive vector $\mathbf{c}$ we have the following isomorphisms

$$X \cong \mathrm{MultiProj}(R) \cong \mathrm{Proj}\big(R^{(\Delta)}\big) \cong \mathrm{Proj}\big(R^{(\mathbf{c})}\big). \tag{3.1}$$

Given a multi-graded R-module M, we get an associated quasi-coherent sheaf $\mathcal{M} = \widetilde{M}$ of $\mathcal{O}_X$-modules. We have the following relations between sheaf and local cohomologies (see e.g. [87], [47, Appendix A4.1]):

(i) There is an exact sequence of multi-graded R-modules

$$0 \to H^0_{\mathfrak{N}}(M) \to M \to \bigoplus_{\mathbf{n}\in\mathbb{Z}^m} H^0(X, \mathcal{M}(\mathbf{n})) \to H^1_{\mathfrak{N}}(M) \to 0. \tag{3.2}$$

(ii) For $j \geqslant 1$, there is an isomorphism of multi-graded R-modules

$$H^{j+1}_{\mathfrak{N}}(M) \cong \bigoplus_{\mathbf{n}\in\mathbb{Z}^m} H^j(X, \mathcal{M}(\mathbf{n})). \tag{3.3}$$

Let I be the multi-homogeneous ideal $I = (f_0, \ldots, f_s)$. Since $\mathcal{F}$ is a dominant rational map, the homogeneous coordinate ring S of Y is often called the special fiber ring in the literature (see (3.5)). Using the canonical graded homomorphism associated to $\mathcal{F}$

$$\begin{array}{rcl} \mathbb{k}[y_0, \ldots, y_s]/\mathfrak{b} & \to & R \\ y_i & \mapsto & f_i, \end{array}$$

we classically identify S with the standard graded $\mathbb{k}$-algebra $\mathbb{k}[I_\mathbf{d}]$, which can be decomposed as

$$S = \mathbb{k}[I_\mathbf{d}] = \bigoplus_{n=0}^{\infty} [I^n]_{n\cdot\mathbf{d}}$$

(see the next subsection for more details).

For the rest of this section we shall assume the following.

Setup 3.2. *Let* $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y \subset \mathbb{P}^s$ *be a dominant rational map defined by* $s+1$ *multi-homogeneous forms* $\mathbf{f} = \{f_0, f_1, \ldots, f_s\} \subset R$ *of the same multi-degree* $\mathbf{d} = (d_1, \ldots, d_m)$. *Let* δ_i *be the dimension* $\delta_i = \dim(X_i)$ *of the projective variety* X_i, *and* $\delta = \delta_1 + \cdots + \delta_m$ *the dimension of* X. *Let* I *be the multi-homogeneous ideal generated by* $f_0, f_1, \ldots, f_s$. *Let* S *be the* *homogeneous coordinate ring* $S = \mathbb{k}[f_0, f_1, \ldots, f_s] = \mathbb{k}[I_{\mathbf{d}}]$ *of* Y *and let* T *be the multi-Veronese subring* $T = R^{(\mathbf{d})} = \mathbb{k}[R_{\mathbf{d}}]$. *After regrading, we regard* $S \subset T$ *as standard graded* $\mathbb{k}$*-algebras.*

The saturated special fiber ring

In this section, we introduce and study an algebra which is the saturated version of the special fiber ring. For any ideal $J \subset R$, the saturation ideal $(J : \mathfrak{N}^\infty)$ with respect to $\mathfrak{N}$ will be written as J^{sat}.

Definition 3.3. *The saturated special fiber ring of* I *is the graded* S*-algebra*

$$\widetilde{\mathfrak{F}_R(I)} = \bigoplus_{n=0}^{\infty} \left[(I^n)^{\text{sat}}\right]_{n \cdot \mathbf{d}}.$$

Interestingly, the algebra $\widetilde{\mathfrak{F}_R(I)}$ turns out to be finitely generated (Lemma 3.8(ii)).

The central point of our approach is a comparison between the two algebras S and $\widetilde{\mathfrak{F}_R(I)}$. Perhaps surprisingly, we show that assuming the condition of S being integrally closed, then $\mathcal{F}$ is birational if and only if these two algebras coincide, and more generally, we show that the difference between their multiplicities yields the degree of $\mathcal{F}$. In addition, we also prove that the algebra $\widetilde{\mathfrak{F}_R(I)}$ reduces the study of the rational map $\mathcal{F}$ to the study of a finite morphism.

Theorem 3.4. *Let* $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y$ *be a dominant rational map. If* $\dim(Y) = \delta$, *then we have the following commutative diagram*

$$\begin{tikzcd}
X \arrow[d, "\cong"'] \arrow[rrr, dashed, "\mathcal{F}"] & & & Y \\
\mathrm{Proj}(T) \arrow[rrr, dashed, "\mathcal{F}'"] \arrow[dr, dashed, "\mathcal{G}"] & & & \\
& \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right) \arrow[urr, "\mathcal{H}"] & &
\end{tikzcd}$$

where the maps $\mathcal{F}' : \mathrm{Proj}(T) \dashrightarrow Y$, $\mathcal{G} : \mathrm{Proj}(T) \dashrightarrow \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right)$ *and* $\mathcal{H} : \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right) \rightarrow Y$ *are induced from the inclusions* $S \hookrightarrow T$, $\widetilde{\mathfrak{F}_R(I)} \hookrightarrow T$ *and* $S \hookrightarrow \widetilde{\mathfrak{F}_R(I)}$, *respectively.*

Also, the statements below are satisfied:

(i) $\mathcal{H} : \mathrm{Proj}\left(\widetilde{\mathfrak{F}_R(I)}\right) \rightarrow Y$ *is a finite morphism with* $\deg(\mathcal{F}) = \deg(\mathcal{H})$.

(ii) $\mathcal{G}$ *is a birational map.*

(iii) $e\left(\widetilde{\mathfrak{F}_R(I)}\right) = \deg(\mathcal{F}) \cdot e(S)$, *where* $e(-)$ *stands for multiplicity.*

(iv) *Under the additional condition of* S *being integrally closed, then* $\mathcal{F}$ *is birational if and only if* $\widetilde{\mathfrak{F}_R(I)} = S$.

The rest of this section is dedicated to the proof of this theorem. Before, we need some intermediate results and definitions. We begin with the following lemma that has its roots in a similar result for the single-graded case (see e.g. [131, Proof of Theorem 6.6], [46, Proof of Proposition 2.11], [102, Remark 2.10]).

Lemma 3.5. *The degree of* $\mathcal{F}$ *is given by* $\deg(\mathcal{F}) = [T : S]$.

Proof. By definition we have that $\deg(\mathcal{F}) = [K(X) : K(Y)] = \left[\left(R^{(\mathbf{d})}\right)_{(0)} : S_{(0)}\right]$. Let $0 \neq f \in I_{\mathbf{d}}$. Then, we have

$$\operatorname{Quot}(S) = S_{(0)}(f) \qquad \text{and} \qquad \operatorname{Quot}(T) = \left(R^{(\mathbf{d})}\right)_{(0)}(f).$$

Finally, since f is transcendental over $\left(R^{(\mathbf{d})}\right)_{(0)}$ and $S_{(0)}$, then it follows that

$$\deg(\mathcal{F}) = \left[\left(R^{(\mathbf{d})}\right)_{(0)} : S_{(0)}\right] = \left[\left(R^{(\mathbf{d})}\right)_{(0)}(f) : S_{(0)}(f)\right] = [T : S],$$

as claimed. □

Now, we introduce a new multi-graded algebra $\mathfrak{A}$ defined by $\mathfrak{A} = R[y_0, y_1, \ldots, y_s]$. By an abuse of notation, for any multi-homogeneous element $g \in \mathfrak{A}$ we will write $\operatorname{bideg}(g) = (\mathbf{a}, b)$ if $\mathbf{a} \in \mathbb{Z}^m$ corresponds with the multi-degree part in R and $b \in \mathbb{Z}$ with the degree part in $\mathbb{k}[\mathbf{y}]$. We give the multi-degrees $\operatorname{bideg}(x_i) = (\deg(x_i), 0)$, $\operatorname{bideg}(y_i) = (\mathbf{0}, 1)$, where $\mathbf{0} \in \mathbb{Z}^m$ denotes a vector $\mathbf{0} = (0, \ldots, 0)$ of m copies of 0.

Given a multi-graded $\mathfrak{A}$-module M and a multi-degree vector $\mathbf{c} \in \mathbb{Z}^m$, then $M_{\mathbf{c}}$ will denote the $\mathbf{c}$-th multi-graded part in R, that is

$$M_{\mathbf{c}} = \bigoplus_{n \in \mathbb{Z}} M_{\mathbf{c},n}.$$

We remark that $M_{\mathbf{c}}$ has a natural structure as a graded $\mathbb{k}[\mathbf{y}]$-module.

We can present the Rees algebra $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n t^n \subset R[t]$ as a quotient of the multi-graded algebra $\mathfrak{A} = R[y_0, y_1, \ldots, y_s]$ via the map

$$\begin{aligned} \Psi : \mathfrak{A} &\longrightarrow \mathcal{R}(I) \subset R[t] \\ y_i &\mapsto f_i t. \end{aligned} \tag{3.4}$$

We set $\operatorname{bideg}(t) = (-\mathbf{d}, 1)$, which implies that Ψ is multi-homogeneous of degree zero. Thus, the multi-graded structure of $\mathcal{R}(I)$ is given by

$$\mathcal{R}(I) = \bigoplus_{\mathbf{c} \in \mathbb{Z}^m, n \in \mathbb{Z}} [\mathcal{R}(I)]_{\mathbf{c},n} \qquad \text{and} \qquad [\mathcal{R}(I)]_{\mathbf{c},n} \cong [I^n]_{\mathbf{c}+n\cdot\mathbf{d}}.$$

We denote $[\mathcal{R}(I)]_{\mathbf{c}} = \bigoplus_{n=0}^{\infty} [\mathcal{R}(I)]_{\mathbf{c},n}$, and of particular interest is the case $[\mathcal{R}(I)]_{\mathbf{0}} \cong \bigoplus_{n=0}^{\infty} [I^n]_{n\cdot\mathbf{d}}$. Let $\mathfrak{M} = \bigoplus_{(c_1,\ldots,c_m)\neq\mathbf{0}} R_{c_1,\ldots,c_m}$, then we have

$$\mathcal{R}(I) = [\mathcal{R}(I)]_{\mathbf{0}} \oplus \left(\bigoplus_{\mathbf{c}\neq\mathbf{0}} [\mathcal{R}(I)]_{\mathbf{c}}\right) = [\mathcal{R}(I)]_{\mathbf{0}} \oplus \mathfrak{M}\mathcal{R}(I).$$

Therefore, we obtain the following isomorphisms

$$S \cong \mathbb{k}\,[I_{\mathbf{d}}] \cong [\mathcal{R}(I)]_{\mathbf{0}} \cong \mathcal{R}(I)/\mathfrak{M}\mathcal{R}(I) \tag{3.5}$$

of graded $\mathbb{k}$-algebras. So, as in Definition 1.4, we may write $S \cong \mathfrak{F}_R(I)$.

We note that each local cohomology module $H_{\mathfrak{N}}^i(\mathcal{R}(I))$ has a natural structure of multi-graded $\mathfrak{A}$-module (see Lemma 2.2), and also that $\left[H_{\mathfrak{N}}^i(\mathcal{R}(I))\right]_{\mathbf{c}} = \bigoplus_{n\in\mathbb{Z}} \left[H_{\mathfrak{N}}^i(I^n)\right]_{\mathbf{c}+n\cdot\mathbf{d}}$ has a natural structure as a graded $\mathbb{k}[\mathbf{y}]$-module. Let $\mathfrak{b} = \mathrm{Ker}(\mathbb{k}[\mathbf{y}] \to \mathbb{k}[\mathbf{f}t])$ be the kernel of the map

$$\mathbb{k}[\mathbf{y}] \to \mathbb{k}[\mathbf{f}t] \subset \mathcal{R}(I), \qquad y_i \mapsto f_i t,$$

then we have that $S \cong \mathbb{k}[\mathbf{y}]/\mathfrak{b}$ and that for any $h \in \mathfrak{b}$ the multiplication map $\mathcal{R}(I) \xrightarrow{\cdot h} \mathcal{R}(I)$ is zero. So the induced map on local cohomology $\left[H_{\mathfrak{N}}^i(\mathcal{R}(I))\right]_{\mathbf{c}} \xrightarrow{\cdot h} \left[H_{\mathfrak{N}}^i(\mathcal{R}(I))\right]_{\mathbf{c}}$ is also zero for any $h \in \mathfrak{b}$. This implies that $\left[H_{\mathfrak{N}}^i(\mathcal{R}(I))\right]_{\mathbf{c}}$ has a natural structure as a graded S-module.

Remark 3.6. *The blow-up $\widetilde{X} = \mathrm{Bl}_I(X)$ of X along $V(I)$ is defined as the multi-projective scheme obtained by considering the Rees algebra $\mathcal{R}(I)$ as a multi-graded $\mathfrak{A}$-algebra. We shall use the notation*

$$\widetilde{X} = \mathrm{MultiProj}_{\mathfrak{A}\text{-gr}}(\mathcal{R}(I)) \subset X \times \mathbb{P}^s,$$

where $\widetilde{X}$ can be canonically embedded in $X \times \mathbb{P}^s$.

By considering the Rees algebra $\mathcal{R}(I)$ only as a multi-graded R-algebra, then we obtain a multi-projective scheme which is an "affine version" of the blow-up $\widetilde{X}$, and that we shall denote by

$$\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)) \subset X \times \mathbb{A}^{s+1},$$

where $\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))$ *can be canonically embedded in* $X \times \mathbb{A}^{s+1}$.

Proposition 3.7. *For each $i \geqslant 0$ and $\mathbf{c} = (c_1, \ldots, c_m)$, we have the following statements:*

(i) $\left[H_{\mathfrak{N}}^i(\mathcal{R}(I))\right]_{\mathbf{c}}$ *is a finitely generated graded S-module.*

(ii) $H^i\left(\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)), \mathcal{O}_{\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))}(\mathbf{c})\right)$ *is a finitely generated graded S-module.*

Proof. (i) It is enough to prove that $\left[H_{\mathfrak{N}}^i(\mathcal{R}(I))\right]_{\mathbf{c}}$ is a finitely generated $\mathbb{k}[\mathbf{y}]$-module. Suppose that $F_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to 0$ represents the minimal free resolution of $\mathcal{R}(I)$ as an $\mathfrak{A}$-module. Let

$\mathcal{C}^\bullet_{\mathfrak{N}}$ be the Čech complex with respect to the ideal $\mathfrak{N}$. We consider the double complex $F_\bullet \otimes_R \mathcal{C}^\bullet_{\mathfrak{N}}$. Computing cohomology by rows and then by columns, gives us a spectral sequence that collapses on the first column and the terms are equal to $H^i_{\mathfrak{N}}(\mathcal{R}(I))$. On the other hand, by computing cohomology by columns, we get the spectral sequence

$$E_1^{-p,q} = H^q_{\mathfrak{N}}(F_p) \Longrightarrow H^{q-p}_{\mathfrak{N}}(\mathcal{R}(I)).$$

Since each $\left[H^q_{\mathfrak{N}}(F_p)\right]_{\mathbf{c}}$ is a finitely generated (free) $\mathbb{k}[\mathbf{y}]$-module, then it follows that $\left[H^{q-p}_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{c}}$ is also finitely generated as a $\mathbb{k}[\mathbf{y}]$-module.

(ii) When $i \geqslant 1$, we get the result from (3.3) and the previous part (i). By (3.2) we have the exact sequence

$$0 \rightarrow [\mathcal{R}(I)]_{\mathbf{c}} \rightarrow H^0\left(\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)), \mathcal{O}_{\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))}(\mathbf{c})\right) \rightarrow \left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{c}} \rightarrow 0.$$

The S-module $[\mathcal{R}(I)]_{\mathbf{c}}$ is clearly finitely generated, and from part (i) the S-module $\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{c}}$ is also finitely generated. Therefore, the exact sequence above gives us the result for $i = 0$. $\square$

By Proposition 3.7, the multi-projective scheme $\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))$ yields the following finitely generated S-algebra

$$\widehat{S} := H^0\left(\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)), \mathcal{O}_{\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))}\right).$$

We will see that $\widehat{S}$ carries the same information as $\widetilde{\mathfrak{F}_R(I)}$, but has the advantage of having some geometrical content as the global sections of an "affine version" of the blow-up $\widetilde{X}$, a fact which is going to be fundamental in the proofs of Theorem 3.4 and Corollary 3.12.

Let $\{\vartheta_1, \ldots, \vartheta_r\}$ be a set of generators of $\mathfrak{N}$, then $\mathrm{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))$ has an affine open cover

$$\mathcal{U} = (U_i)_{i=1,\ldots,r}, \qquad U_i = \mathrm{Spec}\left(\mathcal{R}(I)_{(\vartheta_i)}\right), \tag{3.6}$$

where $\mathcal{R}(I)_{(\vartheta_i)}$ denotes the graded S-module

$$\mathcal{R}(I)_{(\vartheta_i)} = \left[\mathcal{R}(I)_{\vartheta_i}\right]_{\mathbf{0},*}$$

obtained by restricting to multi-degree $\mathbf{0}$ in the multi-grading corresponding with R. Since

$$\mathcal{R}(I) \cong \bigoplus_{n=0}^{\infty} I^n(n \cdot \mathbf{d}),$$

then computing Čech cohomology with respect to $\mathcal{U}$ gives us the following equality

$$\widehat{S} = \bigoplus_{n=0}^{\infty} H^0\left(X, (I^n)^{\sim}(n \cdot \mathbf{d})\right).$$

In the next lemma, with some simple remarks, we show that $\widetilde{\mathfrak{F}_R(I)}$ and $\widehat{S}$ are almost the same.

Lemma 3.8. *The following statements hold:*

(i) There is an inclusion $\widetilde{\mathfrak{F}_R(I)} \subset \widehat{S}$*, which becomes an equality* $\widetilde{\mathfrak{F}_R(I)}_n = \widehat{S}_n$ *for* $n \gg 0$*.*

(ii) $\widetilde{\mathfrak{F}_R(I)}$ *is a finitely generated* S*-module.*

(iii) The two algebras have the same multiplicity, that is $e(\widetilde{\mathfrak{F}_R(I)}) = e(\widehat{S})$*.*

(iv) $\mathrm{Proj}(\widetilde{\mathfrak{F}_R(I)}) \cong \mathrm{Proj}(\widehat{S})$*.*

(v) If $\mathrm{grade}(\mathfrak{N}) \geqslant 2$*, then* $\widetilde{\mathfrak{F}_R(I)} = \widehat{S}$*.*

Proof. (i) Since we have an isomorphism of sheaves $(I^n)^{\sim}(n \cdot \mathbf{d}) \cong \left((I^n)^{\mathrm{sat}}\right)^{\sim}(n \cdot \mathbf{d})$, from (3.2) we get the short exact sequence

$$0 \to \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}} \to H^0\left(X, (I^n)^{\sim}(n \cdot \mathbf{d})\right) \to \left[H^1_{\mathfrak{N}}\left((I^n)^{\mathrm{sat}}\right)\right]_{n\cdot\mathbf{d}} \to 0 \tag{3.7}$$

for each $n \geqslant 1$. The short exact sequence $0 \to (I^n)^{\mathrm{sat}} \to R \to R/(I^n)^{\mathrm{sat}} \to 0$ yields the long exact sequence

$$\left[H^0_{\mathfrak{N}}\left(R/(I^n)^{\mathrm{sat}}\right)\right]_{n\cdot\mathbf{d}} \to \left[H^1_{\mathfrak{N}}\left((I^n)^{\mathrm{sat}}\right)\right]_{n\cdot\mathbf{d}} \to \left[H^1_{\mathfrak{N}}(R)\right]_{n\cdot\mathbf{d}} \to \left[H^1_{\mathfrak{N}}\left(R/(I^n)^{\mathrm{sat}}\right)\right]_{n\cdot\mathbf{d}}. \tag{3.8}$$

We always have that $H^0_{\mathfrak{N}}\left(R/(I^n)^{\mathrm{sat}}\right) = 0$, and that $\left[H^1_{\mathfrak{N}}(R)\right]_{n\cdot\mathbf{d}} = 0$ for $n \gg 0$ (see e.g. [98, Lemma 4.2]). Hence, we get that $\left[H^1_{\mathfrak{N}}\left((I^n)^{\mathrm{sat}}\right)\right]_{n\cdot\mathbf{d}} = 0$ for $n \gg 0$, and so the result follows.

(ii) Straightforward from part (i) and Proposition 3.7(ii).

(iii) It is clear from part (i).

(iv) Follows from part (i) (see e.g. [66, Exercise II.2.14]).

(v) The condition $\mathrm{grade}(\mathfrak{N}) \geqslant 2$ implies that $H^1_{\mathfrak{N}}(R) = 0$, then the required equality is obtained from (3.7) and (3.8). ☐

All the rational maps that are considered in specific applications are usually such that $\mathrm{grade}(\mathfrak{N}) \geqslant 2$. So, in practice, we always have $\widetilde{\mathfrak{F}_R(I)} = \widehat{S}$. Nevertheless, we give an example where $\widetilde{\mathfrak{F}_R(I)}$ and $\widehat{S}$ are different.

Example 3.9. *Let* $R = \Bbbk\left[s^4, s^3t, st^3, t^4\right]$ *be the homogeneous coordinate ring of the rational quartic* $\mathcal{C}$ *in the three projective space parametrized by*

$$\mathbb{P}^1 \to \mathcal{C} \subset \mathbb{P}^3, \qquad (s:t) \mapsto (s^4 : s^3t : st^3 : t^4).$$

Computing directly with Macaulay2 [60] *or using* [47, Exercise 18.8], *we get the isomorphism*

$$R \cong \frac{\Bbbk[x_0, x_1, x_2, x_3]}{\left(x_1x_2 - x_0x_3, x_2^3 - x_1x_3^2, x_0x_2^2 - x_1^2x_3, x_1^3 - x_0^2x_2\right)}$$

and that R *is not Cohen-Macaulay with* $\dim(R) = 2$ *and* $\operatorname{depth}(R) = 1$. *Moreover, by setting* $B = \Bbbk[x_0, x_1, x_2, x_3]$ *and* $\mathfrak{m} = (x_0, x_1, x_2, x_3) \subset B$, *further computations with Macaulay2* [60] *show that*

$$\operatorname{Ext}_B^3\left(R, B(-4)\right) \cong \left(B/\mathfrak{m}\right)(1),$$

and so the graded local duality theorem (see e.g. [19, Theorem 3.6.19]*) yields the isomorphism*

$$H_{\mathfrak{m}}^1(R) \cong \left(B/\mathfrak{m}\right)(-1).$$

Let $I = (x_0, x_1, x_3) \subset R$ *be the ideal corresponding with the morphism*

$$\mathcal{C} = \operatorname{Proj}(R) \xrightarrow{(x_0:x_1:x_3)} \mathbb{P}^2.$$

Since we have $I^{\text{sat}} = R$, *then our previous computation gives*

$$\left[H_{\mathfrak{m}R}^1\left(I^{\text{sat}}\right)\right]_1 \cong \left[H_{\mathfrak{m}}^1\left(I^{\text{sat}}\right)\right]_1 = \left[H_{\mathfrak{m}}^1(R)\right]_1 \cong \left[\left(B/\mathfrak{m}\right)(-1)\right]_1 \cong \Bbbk \neq 0.$$

Finally, from (3.7) we obtain that $\widetilde{\mathfrak{F}_R(I)}_1 \neq \widehat{S}_1$.

We are now ready to give the proof of the main result of this section.

Proof of Theorem 3.4. From our previous discussions (3.1), we have the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "\cong"'] \arrow[dr, dashed, "\mathcal{F}"] & \\ \operatorname{Proj}(T) \arrow[r, dashed, "\mathcal{F}'"'] & Y \end{tikzcd}$$

The rational map $\mathcal{F}'$ can be given by the tuple $(f_0 : f_1 : \cdots : f_s)$ because each $f_i \in T$. From a geometrical point of view, here we are embedding X in the "right" projective space

$$\mathbb{P}^M \quad \text{of dimension} \quad M = \prod_{i=1}^{m} \binom{r_i + d_i}{r_i} - 1,$$

where the f_i's are actually linear forms. Then the rational map $\mathcal{F}' : \mathrm{Proj}(T) \dashrightarrow Y = \mathrm{Proj}(S)$ is induced from the canonical inclusion $S \hookrightarrow T$. Since we have the canonical inclusions

$$S \hookrightarrow \widetilde{\mathfrak{F}_R(I)} \hookrightarrow T$$

then $\mathcal{F}'$ is given by the composition of the rational maps

$$\mathrm{Proj}(T) \dashrightarrow \mathrm{Proj}(\widetilde{\mathfrak{F}_R(I)}) \dashrightarrow \mathrm{Proj}(S).$$

From the condition $\dim(X) = \dim(Y)$, we have that

$$\mathrm{Quot}(S) \subset \mathrm{Quot}(\widetilde{\mathfrak{F}_R(I)}) \subset \mathrm{Quot}(T)$$

are algebraic extensions. Therefore, by using the same argument of Lemma 3.5, we get the equalities $\deg(\mathcal{F}) = [T : S]$, $\deg(\mathcal{G}) = [T : \widetilde{\mathfrak{F}_R(I)}]$ and $\deg(\mathcal{H}) = [\widetilde{\mathfrak{F}_R(I)} : S]$.

(i) First we check that the rational map $\mathcal{H} : \mathrm{Proj}(\widetilde{\mathfrak{F}_R(I)}) \dashrightarrow Y$ is actually a finite morphism. Since $\widetilde{\mathfrak{F}_R(I)}$ is a finitely generated S-module (Lemma 3.8(ii)), we even have that $\widetilde{\mathfrak{F}_R(I)}$ is integral over S. By the Incomparability Theorem (see e.g. [47, Corollary 4.18]), the inclusion $S \hookrightarrow \widetilde{\mathfrak{F}_R(I)}$ induces a (well defined everywhere) morphism

$$\mathcal{H} : \mathrm{Proj}(\widetilde{\mathfrak{F}_R(I)}) \longrightarrow \mathrm{Proj}(S).$$

Indeed, for any $\mathfrak{q} \subsetneq \left[\widetilde{\mathfrak{F}_R(I)}\right]_+ = \bigoplus_{c>0} \left[\widetilde{\mathfrak{F}_R(I)}\right]_c$ we necessarily have that

$$\mathfrak{q} \cap S \subsetneq [S]_+ = \bigoplus_{c>0} [S]_c.$$

The finiteness of $\widetilde{\mathfrak{F}_R(I)}$ as an S-module yields that $\mathcal{H}$ is a finite morphism.

Next we will prove that $\deg(\mathcal{H}) = \deg(\mathcal{F})$. Let us denote by $\widetilde{X} = \mathrm{MultiProj}_{\mathfrak{A}\text{-gr}}(\mathcal{R}(I))$ the blow-up of X along $\mathcal{B} = V(I)$, which can also be seen as the closure of the graph of $\mathcal{F}$. We then have the commutative diagram

$$\begin{tikzcd}
\widetilde{X} \subset X \times \mathbb{P}^s \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \arrow[r, dashed, "\mathcal{F}"] & Y \subset \mathbb{P}^s
\end{tikzcd}$$

with π_1 being an isomorphism outside $\mathcal{B}$ (see e.g. [23, Proposition 2.3], [66, Proposition II.7.13]).

Let ξ be the generic point of Y and consider the fiber product $W := \widetilde{X} \times_Y \mathrm{Spec}\,(\mathcal{O}_{Y,\xi})$. Denoting η as the generic point of $\widetilde{X}$, since π_2 is assumed to be generically finite, then we have the isomorphism $\mathrm{Spec}\left(\mathcal{O}_{\widetilde{X},\eta}\right) \cong W$; this is a classical result, for a detailed proof see [137, Tag 02NV].

Even though W is just a point, we will consider a convenient (and trivial) affine open cover of it. The scheme Y has an affine open cover given by $Y_j = \mathrm{Spec}\left(S_{(y_j)}\right)$ for $j = 0, \ldots, s$. The open set $\pi_2^{-1}(Y_j)$ is isomorphic to $\mathrm{MultiProj}_{R\text{-gr}}\left(\mathcal{R}(I)_{(y_j)}\right)$, where $\mathcal{R}(I)_{(y_j)}$ denotes the multi-graded R-module

$$\mathcal{R}(I)_{(y_j)} = \left[\mathcal{R}(I)_{y_j}\right]_{*,0}$$

defined by restricting to elements of degree zero in the grading corresponding with S. Then W can be obtained by glueing the open cover

$$W_j \;:=\; \mathrm{MultiProj}_{R\text{-gr}}\left(\mathcal{R}(I)_{(y_j)}\right) \;\times_{\mathrm{Spec}\left(S_{(y_j)}\right)}\; \mathrm{Spec}\,(\mathcal{O}_{Y,\xi})$$

for $j = 0, \ldots, s$.

Fix $0 \leqslant j \leqslant s$. Similarly to (3.6), the scheme $\mathrm{MultiProj}_{R\text{-gr}}\left(\mathcal{R}(I)_{(y_j)}\right)$ has an affine open cover

$$\left(\mathrm{Spec}\left(\mathcal{R}(I)_{(\vartheta_i y_j)}\right)\right)_{i=1,\ldots,r},$$

where we are using the similar notation $\mathcal{R}(I)_{(\vartheta_i y_j)} = \left[\mathcal{R}(I)_{\vartheta_i y_j}\right]_{\mathbf{0},0}$. Now, W_j is obtained by glueing the affine open cover given by

$$\left(\mathrm{Spec}\left(\mathcal{R}(I)_{(\vartheta_i y_j)} \;\otimes_{S_{(y_j)}}\; \mathcal{O}_{Y,\xi}\right)\right)_{i=1,\ldots,r}. \tag{3.9}$$

Since $\mathcal{O}_{Y,\xi} \cong S_{(0)}$, then we have that the ring $\mathcal{R}(I)_{(\vartheta_i y_j)} \;\otimes_{S_{(y_j)}}\; \mathcal{O}_{Y,\xi}$ does not depend on j. Therefore we obtain that $W = W_j$.

Let K be the multiplicative set of homogeneous elements of S and B be the localized ring $B = K^{-1}S$. We denote by $\mathcal{W} = (\mathcal{W}_i)_{i=1,\ldots,r}$ the affine open cover of (3.9). Since we have the following isomorphisms of multi-graded $\mathfrak{A}$-modules

$$\mathcal{R}(I)_{\vartheta_{i_1}\vartheta_{i_2}\cdots\vartheta_{i_t} y_j} \otimes_{S_{y_j}} B \;\cong\; \mathcal{R}(I)_{\vartheta_{i_1}\vartheta_{i_2}\cdots\vartheta_{i_t}} \otimes_S B \qquad \text{for any } 1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant r,$$

the corresponding Čech complex is given by

$$C^\bullet(\mathcal{W}): \qquad 0 \to \bigoplus_i \mathcal{R}(I)_{\vartheta_i} \otimes_S B \to \bigoplus_{i<k} \mathcal{R}(I)_{\vartheta_i\vartheta_k} \otimes_S B \to \cdots \to \mathcal{R}(I)_{\vartheta_1\cdots\vartheta_r} \otimes_S B \to 0.$$

Using the affine open cover (3.6) of $\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))$, we get the similar Čech complex

$$C^\bullet(\mathcal{U}): \qquad 0 \to \bigoplus_i \mathcal{R}(I)_{\vartheta_i} \to \bigoplus_{i<k} \mathcal{R}(I)_{\vartheta_i\vartheta_k} \to \cdots \to \mathcal{R}(I)_{\vartheta_1\cdots\vartheta_r} \to 0.$$

Since B is flat over S, we get the isomorphism of multi-graded $\mathfrak{A}$-modules

$$H^0(C^\bullet(\mathcal{U})) \otimes_S B \;\cong\; H^0(C^\bullet(\mathcal{W})),$$

and restricting to the multi-degree $\mathbf{0}$ part in R, we get the following isomorphisms of graded S-modules

$$\begin{aligned}\widehat{S} \otimes_S B &= H^0\left(\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)), \mathcal{O}_{\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))}\right) \otimes_S B \\ &\cong \left[H^0(C^\bullet(\mathcal{U}))\right]_{\mathbf{0}} \otimes_S B \;\cong\; \left[H^0(C^\bullet(\mathcal{W}))\right]_{\mathbf{0}}.\end{aligned}$$

From the fact that $S \hookrightarrow \widehat{S}$ is an algebraic extension, we have $\text{Quot}(\widehat{S}) = \widehat{S} \otimes_S \text{Quot}(S)$. So, by restricting to the degree zero part, we get the following isomorphisms of rings

$$\widehat{S}_{(0)} = \left[\widehat{S} \otimes_S B\right]_0 \cong \left[\left[H^0(C^\bullet(\mathcal{U}))\right]_{\mathbf{0}} \otimes_S B\right]_0 \cong \left[H^0(C^\bullet(\mathcal{W}))\right]_{\mathbf{0},0} \cong H^0(W, \mathcal{O}_W) = \mathcal{O}_{\widetilde{X},\eta}.$$

Finally, since π_1 is a birational morphism and $\widetilde{\mathfrak{F}_R(I)}_{(0)} = \widehat{S}_{(0)}$ (Lemma 3.8(iv)), we obtain

$$\deg(\mathcal{H}) = \left[\widetilde{\mathfrak{F}_R(I)}_{(0)} : S_{(0)}\right] = \left[\widehat{S}_{(0)} : S_{(0)}\right] = \left[\mathcal{O}_{\widetilde{X},\eta} : \mathcal{O}_{Y,\xi}\right] = \deg(\pi_2) = \deg(\mathcal{F}).$$

(ii) From part (i) we have $\deg(\mathcal{F}) = \deg(\mathcal{H})$. Then, the equality $\deg(\mathcal{F}) = \deg(\mathcal{G})\deg(\mathcal{H})$ gives us that $\deg(\mathcal{G}) = 1$.

(iii) From the associative formula for multiplicity [19, Corollary 4.6.9] and part (i), we get

$$e(\widetilde{\mathfrak{F}_R(I)}) = \left[\widetilde{\mathfrak{F}_R(I)} : S\right] e(S) = \deg(\mathcal{F})e(S).$$

(iv) We only need to prove that assuming the birationality of $\mathcal{F}$ and that S is integrally closed, then we get $\widetilde{\mathfrak{F}_R(I)} = S$. The equality $\deg(\mathcal{F}) = \deg(\mathcal{H}) = \left[\widetilde{\mathfrak{F}_R(I)} : S\right]$ and the birationality of $\mathcal{F}$ imply that $\text{Quot}(\widetilde{\mathfrak{F}_R(I)}) = \text{Quot}(S)$. Therefore we have the following canonical inclusions

$$S \subset \widetilde{\mathfrak{F}_R(I)} \subset \text{Quot}(\widetilde{\mathfrak{F}_R(I)}) = \text{Quot}(S),$$

and so from the fact that S is integrally closed and that $S \hookrightarrow \widetilde{\mathfrak{F}_R(I)}$ is an integral extension, we obtain $\widetilde{\mathfrak{F}_R(I)} = S$. □

We end this subsection by providing a relation between the j-multiplicity of an ideal and the multiplicity of the corresponding saturated special fiber ring. The j-multiplicity of an ideal was introduced in [1]. It serves as a generalization of the Hilbert-Samuel multiplicity, and has applications in intersection theory (see e.g. [53]).

Let A be a standard graded $\mathbb{k}$-algebra of dimension $\delta + 1$ which is an integral domain. Let $\mathfrak{n}$ be its maximal irrelevant ideal $\mathfrak{n} = A_+$. For a non necessarily $\mathfrak{n}$-primary ideal $J \subset A$ its j-multiplicity is given by

$$j(J) = \delta! \lim_{n\to\infty} \frac{\dim_{\mathbb{k}}\left(H^0_{\mathfrak{n}}\left(J^n/J^{n+1}\right)\right)}{n^\delta}.$$

Lemma 3.10. *Let $J \subset A$ be a homogeneous ideal equally generated in degree d. Suppose J has maximal analytic spread $\ell(J) = \delta + 1$. Then, we have the equality*

$$j(J) = d \cdot e\left(\widetilde{\mathfrak{F}_A(J)}\right),$$

where $\widetilde{\mathfrak{F}_A(J)} = \bigoplus_{n=0}^\infty \left[(J^n)^{\text{sat}}\right]_{nd}$ is the saturated special fiber ring of J.

Proof. We consider the associated dominant rational map $\mathcal{G} : \text{Proj}\,(A) \dashrightarrow \text{Proj}\,(\mathbb{k}[J_d])$, that satisfies $\dim(A) = \dim(\mathbb{k}[J_d])$ because $\ell(J) = \delta + 1$. From [102, Theorem 5.3] and Theorem 3.4(iii) we obtain

$$j(J) = d \cdot \deg(\mathcal{G}) \cdot e\left(\mathbb{k}[J_d]\right) \quad \text{and} \quad e\left(\widetilde{\mathfrak{F}_A(J)}\right) = \deg(\mathcal{G}) \cdot e\left(\mathbb{k}[J_d]\right),$$

respectively. So the result follows by comparing both equations. □

As a direct consequence of this lemma we obtain a refined version of [92, Theorem 3.1(iii)].

Corollary 3.11. *Let $J \subset A$ be a homogeneous ideal equally generated in degree d. Suppose J has maximal analytic spread $\ell(J) = \delta + 1$. If $\left[(J^n)^{\text{sat}}\right]_{nd} = [J^n]_{nd}$ for all $n \gg 0$, then*

$$j(J) = d \cdot e(\mathbb{k}[J_d]).$$

Formula for the degree of multi-graded rational maps

In this subsection, we prove a new formula that relates the degree of $\mathcal{F}$ with the multiplicity of the S-module $\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}$ and the degree of the image. This result will be our main tool for making specific computations. To state it, we will need the following additional notation: for any finitely generated graded S-module N, the $(\delta + 1)$-th multiplicity is defined by (see e.g. [19, §4.7])

$$e_{\delta+1}(N) = \begin{cases} e(N) & \text{if } \dim(N) = \delta + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.12. *Let $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y$ be a dominant rational map. If* $\dim(Y) = \delta$, *then the degree of $\mathcal{F}$ can be computed by*

$$\deg_{\mathbb{P}^s}(Y)(\deg(\mathcal{F}) - 1) = e_{\delta+1}\left(\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\right) = \delta! \lim_{n\to\infty} \frac{\dim_{\mathbb{k}}\left(\left[H^1_{\mathfrak{N}}(I^n)\right]_{n\cdot\mathbf{d}}\right)}{n^\delta}.$$

In particular, we have that $\mathcal{F}$ is birational if and only if $\dim_S\left(\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\right) < \delta + 1$.

Proof. From (3.2) we have the exact sequence

$$0 \to [\mathcal{R}(I)]_{\mathbf{0}} \to H^0\left(\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)), \mathcal{O}_{\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))}\right) \to \left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}} \to 0$$

which by using our previous notations can be written as

$$0 \to S \to \widehat{S} \to \left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}} \to 0.$$

We clearly have $e_{\delta+1}(S) = \deg_{\mathbb{P}^s}(Y)$, then it follows that

$$\begin{aligned} e_{\delta+1}(\widehat{S}) &= e_{\delta+1}(\widetilde{\mathfrak{F}_R(I)}) && \text{(by Lemma 3.8(iii))} \\ &= \deg(\mathcal{F}) \cdot e_{\delta+1}(S) && \text{(by Theorem 3.4(iii))} \\ &= \deg(\mathcal{F}) \cdot \deg_{\mathbb{P}^s}(Y). \end{aligned}$$

Therefore, the previous exact sequence yields the equality

$$e_{\delta+1}\left(\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\right) = e_{\delta+1}(\widehat{S}) - e_{\delta+1}(S) = \deg_{\mathbb{P}^s}(Y)(\deg(\mathcal{F}) - 1),$$

as claimed. □

Remark 3.13. *Let J be an ideal in the polynomial ring* $\mathbb{k}[x_1, \ldots, x_p]$, *and $\mathfrak{m}$ the maximal irrelevant ideal* $(x_1, \ldots, x_p)$. *In* [43], *it was shown that the limit*

$$\lim_{n\to\infty} \frac{\lambda(H^1_{\mathfrak{m}}(J^n))}{n^p} = \lim_{n\to\infty} \frac{\lambda(H^0_{\mathfrak{m}}(R/J^n))}{n^p}$$

always exists under the assumption that $\mathbb{k}$ is a field of characteristic zero, but, interestingly, it is proved that it is not necessarily a rational number. Later, in [74] *it was obtained that when J is a monomial ideal this limit is a rational number. From the previous Corollary 3.12 we have that a similar limit obtained by restricting to certain graded strands, is always rational and also can give valuable information for a (multi-graded) rational map.*

3.2 Rational maps with zero-dimensional base locus

In this section we restrict ourselves to the case where the base locus $\mathcal{B} = V(I)$ has dimension zero, i.e. that $\mathcal{B}$ is finite over $\mathbb{k}$. In this case, we obtain four main different lines of results, that we gather in four subsections. Firstly, in §4.2.1, we provide an algebraic proof of the degree formula in the general multi-graded case. Then, in §4.2.2, we derive bounds for the degree of a rational map from Corollary 3.12, in terms of the symmetric algebra. Thirdly, in §4.2.3, we apply our methods in the case of rational maps defined over multi-projective spaces. And we conclude by providing an upper bound for the degree of a single-graded rational map in terms of certain values of the Hilbert function of the base ideal in §4.2.4.

We shall see that these upper bounds are sharp in some cases, and also that we obtain new effective birationality criteria under certain conditions.

The degree formula

We give a formula for the degree of a multi-graded rational map, which depends on the degrees of the source and the image, and the multiplicity of the base points. This known formula can also be obtained with more geometric techniques (see [55, Section 4.4]). It can be seen as a generalization of the same result in the single-graded case (see [23, Theorem 2.5] and [131, Theorem 6.6]). Hereafter we use the same notations and conventions of §4.1.1. We begin with two preliminary results.

Proposition 3.14. *Assume that* $\mathcal{F} : X = X_1 \times \cdots \times X_m \dashrightarrow Y$ *is generically finite. Then, we have that* $\dim_S\left(\left[H^i_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\right) < \dim(S)$ *for all* $i \geqslant 2$.

Proof. We have defined $\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I))$ by considering $\mathcal{R}(I)$ as a multi-graded R-algebra, and so we have the following morphisms

$$\pi_2 : \text{MultiProj}_{\mathfrak{A}\text{-gr}}(\mathcal{R}(I)) \subset X \times \mathbb{P}^s \longrightarrow \text{Proj}(S) \subset \mathbb{P}^s$$
$$\nu : \text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)) \subset X \times \mathbb{A}^{s+1} \longrightarrow \text{Spec}(S) \subset \mathbb{A}^{s+1}$$

where both π_2 and ν are determined by the inclusion $S = \mathbb{k}[\mathbf{y}]/\mathfrak{b} \hookrightarrow \mathcal{R}(I)$ that sends y_i into $f_i t$, and the only difference consists on whether we take into account the grading in $\mathbf{y}$ or not. Therefore, we have that ν is also generically finite, and there exists some $L \in S$ for which the morphism

$$\nu_L : \text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)_L) \to \text{Spec}(S_L)$$

is finite (see [66, Exercise II.3.7]). Thus, it follows that $\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)_L)$ is affine (see [66, Exercise II.5.17]).

From the vanishing of sheaf cohomology (see [66, III.3]) and (3.3), we get

$$\left(\left[H^i_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\right)_L \cong \left[H^i_{\mathfrak{N}}(\mathcal{R}(I)_L)\right]_{\mathbf{0}} \cong H^{i-1}\left(\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)_L), \mathcal{O}_{\text{MultiProj}_{R\text{-gr}}(\mathcal{R}(I)_L)}\right) = 0$$

for all $i \geqslant 2$. Since $\left[H^i_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}$ is a finitely generated graded S-module then it is annihilated by some power of L, and the claimed result follows. □

We define the degree of X as the degree of its corresponding projectively embedded variety in $\mathbb{P}^N$ by means of the Segre embedding. We have the following relation between the degree of X and the degrees of the projective varieties $X_i \subset \mathbb{P}^{r_i}$, $i = 1, \ldots, m$.

Lemma 3.15. *The degree of* $X = X_1 \times \cdots \times X_m$ *can be computed as*

$$\deg_{\mathbb{P}^N}(X) = \frac{\delta!}{\delta_1!\delta_2!\cdots\delta_m!}\deg_{\mathbb{P}^{r_1}}(X_1)\deg_{\mathbb{P}^{r_2}}(X_2)\cdots\deg_{\mathbb{P}^{r_m}}(X_m).$$

Proof. Since the homogeneous coordinate ring of the image of X in the Segre embedding is given by $R^{(\Delta)}$, we have the following equality

$$P_{R^{(\Delta)}}(t) = P_{A_1}(t)P_{A_2}(t)\cdots P_{A_m}(t)$$

between the Hilbert polynomials of the standard graded $\mathbb{k}$-algebras $A_1, \ldots, A_m$ and $R^{(\Delta)}$. By comparing the leading terms of both sides of the equation we get the claimed result. □

Under the present condition $\dim(\mathcal{B}) = 0$, we define the multiplicity of $\mathcal{B}$ in X by the following formula

$$e(\mathcal{B}) := \delta! \lim_{n\to\infty} \frac{\dim_{\mathbb{k}}\left(H^0\left(X, \mathcal{O}_X/(I^n)^{\sim}\right)\right)}{n^{\delta}}. \tag{3.10}$$

Since we have the equalities

$$\begin{aligned}\dim_{\mathbb{k}}\left(H^0\left(X, \mathcal{O}_X/(I^n)^{\sim}\right)\right) &= \sum_{\mathfrak{p}\in\mathcal{B}} \dim_{\mathbb{k}}\left(\left(\mathcal{O}_X/(I^n)^{\sim}\right)_{\mathfrak{p}}\right)\\ &= \sum_{\mathfrak{p}\in\mathcal{B}} [\mathbb{k}(\mathfrak{p}):\mathbb{k}]\cdot \mathrm{length}_{\mathcal{O}_{X,\mathfrak{p}}}\left(\left(\mathcal{O}_X/(I^n)^{\sim}\right)_{\mathfrak{p}}\right)\\ &= \sum_{\mathfrak{p}\in\mathcal{B}} [\mathbb{k}(\mathfrak{p}):\mathbb{k}]\cdot \mathrm{length}_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/I_{\mathfrak{p}}^n\right),\end{aligned}$$

the expression $\dim_{\mathbb{k}}\left(H^0\left(X, \mathcal{O}_X/(I^n)^{\sim}\right)\right)$ becomes a polynomial for $n \gg 0$. Also, we can compute (3.10) with the following equation

$$e(\mathcal{B}) = \sum_{\mathfrak{p}\in\mathcal{B}} [\mathbb{k}(\mathfrak{p}):\mathbb{k}]\cdot e_{I_{\mathfrak{p}}}(R_{\mathfrak{p}}),$$

where $e_{I_{\mathfrak{p}}}(R_{\mathfrak{p}})$ denotes the Hilbert-Samuel multiplicity of the local ring $R_{\mathfrak{p}}$ with respect to the $\mathfrak{p}R_{\mathfrak{p}}$-primary ideal $I_{\mathfrak{p}}$ (see [19, Section 4.5]).

The degree of the base locus $\mathcal{B} = V(I)$ is defined in a similar way to its multiplicity (3.10). When $\dim(\mathcal{B}) = 0$, $\deg(\mathcal{B})$ is given by the formula

$$\begin{aligned}\deg(\mathcal{B}) := \dim_{\Bbbk}\left(H^0\left(X, \mathcal{O}_X/I^{\sim}\right)\right) &= \sum_{\mathfrak{p}\in\mathcal{B}} \dim_{\Bbbk}\left(\left(\mathcal{O}_X/I^{\sim}\right)_{\mathfrak{p}}\right) \\ &= \sum_{\mathfrak{p}\in\mathcal{B}} [\Bbbk(\mathfrak{p}) : \Bbbk] \cdot \text{length}_{\mathcal{O}_{X,\mathfrak{p}}}\left(\left(\mathcal{O}_X/I^{\sim}\right)_{\mathfrak{p}}\right) \\ &= \sum_{\mathfrak{p}\in\mathcal{B}} [\Bbbk(\mathfrak{p}) : \Bbbk] \cdot \text{length}_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/I_{\mathfrak{p}}\right).\end{aligned}$$

The theorem below provides a new algebraic proof of the degree formula for a multi-graded rational map with finitely many base points.

Theorem 3.16. *Let* $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y$ *be a dominant rational map. If* $\dim(Y) = \delta$ *and* $\dim(\mathcal{B}) = 0$, *then*

$$d_1^{\delta_1} \cdots d_m^{\delta_m} \deg_{\mathbb{P}^N}(X) = \deg_{\mathbb{P}^s}(Y) \deg(\mathcal{F}) + e(\mathcal{B}),$$

or equivalently

$$d_1^{\delta_1} \cdots d_m^{\delta_m} \frac{\delta!}{\delta_1! \cdots \delta_m!} \deg_{\mathbb{P}^{r_1}}(X_1) \cdots \deg_{\mathbb{P}^{r_m}}(X_m) = \deg_{\mathbb{P}^s}(Y) \deg(\mathcal{F}) + e(\mathcal{B}).$$

Proof. For $n \geqslant 1$ we have the exact sequence of sheaves

$$0 \to (I^n)^{\sim}(n \cdot \mathbf{d}) \to \mathcal{O}_X(n \cdot \mathbf{d}) \to \frac{\mathcal{O}_X}{(I^n)^{\sim}}(n \cdot \mathbf{d}) \to 0,$$

that gives us the following equation relating Euler characteristics:

$$\chi\left(X, (I^n)^{\sim}(n \cdot \mathbf{d})\right) + \chi\left(X, \frac{\mathcal{O}_X}{(I^n)^{\sim}}(n \cdot \mathbf{d})\right) = \chi(X, \mathcal{O}_X(n \cdot \mathbf{d})).$$

The term $\chi(X, \mathcal{O}_X(n \cdot \mathbf{d}))$ for sufficiently large n becomes

$$\chi(X, \mathcal{O}_X(n{\cdot}\mathbf{d})) = \dim_{\Bbbk}\left(H^0(X, \mathcal{O}_X(n{\cdot}\mathbf{d}))\right) = \dim_{\Bbbk}(R_{n\cdot\mathbf{d}}) = \frac{d_1^{\delta_1} \cdots d_m^{\delta_m} \deg_{\mathbb{P}^N}(X)}{\delta!} n^{\delta} + \text{lower terms}$$

the Hilbert polynomial of the standard graded $\Bbbk$-algebra $T(= R^{(\mathbf{d})})$ (recall that $H^i(X, \mathcal{O}_X(n \cdot \mathbf{d})) = 0$ for $i \geqslant 1$ and $n \gg 0$; see [87, Theorem 1.6]).

Since $\dim(\mathcal{B}) = 0$, the summand $\chi\Big(X, \frac{\mathcal{O}_X}{(I^n)^\sim}(n \cdot \mathbf{d})\Big)$ for all $n \gg 0$ is a polynomial

$$\chi\Big(X, \frac{\mathcal{O}_X}{(I^n)^\sim}(n \cdot \mathbf{d})\Big) = \dim_{\mathbb{k}}\Big(H^0\Big(X, \frac{\mathcal{O}_X}{(I^n)^\sim}\Big)\Big) = \frac{e(\mathcal{B})}{\delta!} n^\delta + \text{lower terms}$$

whose leading coefficient is equal to the multiplicity of the base points.

We clearly have that $\mathcal{F}$ is a generically finite map, so Proposition 3.14 yields that for any $i \geqslant 1$ and $n \gg 0$, the expression

$$\dim_{\mathbb{k}}\left(H^i(X, (I^n)^\sim(n \cdot \mathbf{d}))\right) = \dim_{\mathbb{k}}\left(\left[H^{i+1}_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0},n}\right)$$

becomes a polynomial of degree strictly less than δ. This implies that the leading coefficient of the polynomial determined by $\chi\big(X, (I^n)^\sim(n \cdot \mathbf{d})\big)$ coincides with the one of the polynomial determined by

$$\dim_{\mathbb{k}}\left(H^0(X, (I^n)^\sim(n \cdot \mathbf{d}))\right).$$

From Theorem 3.4(iii), for $n \gg 0$ the function $\chi\big(X, (I^n)^\sim(n \cdot \mathbf{d})\big)$ is also a polynomial that has the form

$$\chi\big(X, (I^n)^\sim(n \cdot \mathbf{d})\big) = \frac{\deg_{\mathbb{P}^s}(Y)\deg(\mathcal{F})}{\delta!} n^\delta + \text{lower terms.}$$

Finally, comparing the leading coefficients of these polynomials, the equation

$$d_1^{\delta_1} \cdots d_m^{\delta_m} \deg_{\mathbb{P}^N}(X) = \deg_{\mathbb{P}^s}(Y)\deg(\mathcal{F}) + e(\mathcal{B})$$

follows. The other formula is equivalent from Lemma 3.15. $\square$

Degree and syzygies of the base ideal

In this subsection, using the close link between Rees and symmetric algebras, we derive some consequences of Corollary 3.12 in terms of the symmetric algebra of the base ideal of a rational map. Under the assumption of having a zero dimensional base locus, we bound the multiplicity $e_{\delta+1}\left(\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\right)$ of the Rees algebra with the corresponding multiplicity $e_{\delta+1}\left(\left[H^1_{\mathfrak{N}}(\mathrm{Sym}(I))\right]_{\mathbf{0}}\right)$ of the symmetric algebra, and the later one is bounded by using the $\mathcal{Z}_\bullet$ approximation complex.

We keep here similar notations with respect to the previous one, but we assume that the image Y is the projective space $\mathbb{P}^\delta$. We take this assumption because in general the symmetric algebra $\mathrm{Sym}(I)$ is only a $\mathbb{k}[\mathbf{y}]$-module and not an S-module (Setup 3.2). To be precise, we restate the notations that we use in this subsection.

Setup 3.17. *Let* $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow \mathbb{P}^\delta$ *be a dominant rational map defined by* $\delta+1$ *multi-homogeneous forms* $\mathbf{f} = \{f_0, f_1, \ldots, f_\delta\} \subset R$ *of the same multi-degree* $\mathbf{d} = (d_1, \ldots, d_m)$*, where* δ *is the dimension of* X*. Let* I *be the multi-homogeneous ideal generated by* $f_0, f_1, \ldots, f_\delta$*. Let* S *be the homogeneous coordinate ring* $S = \mathbb{k}[y_0, y_1, \ldots, y_\delta]$ *of* $\mathbb{P}^\delta$*.*

Remark 3.18. *Given a finitely generated* S*-module* N*, from the associative formula for multiplicity* [19, Corollary 4.6.9]*, we get*

$$e_{\delta+1}(N) = \operatorname{rank}(N).$$

The Rees algebra $\mathcal{R}(I)$ has a natural structure of multi-graded $\mathfrak{A}$-module by (3.4). Also, from the minimal graded presentation of I

$$F_1 \xrightarrow{\varphi} F_0 \xrightarrow{(f_0,\ldots,f_\delta)} I \to 0,$$

the symmetric algebra

$$\mathrm{Sym}(I) \cong \mathfrak{A}/I_1\left((y_0,\cdots,y_\delta)\cdot\varphi\right)$$

has a natural structure of multi-graded $\mathfrak{A}$-module. Therefore, we have a canonical exact sequence of multi-graded $\mathfrak{A}$-modules relating both algebras

$$0 \to \mathcal{K} \to \mathrm{Sym}(I) \to \mathcal{R}(I) \to 0. \tag{3.11}$$

The following result is likely part of the folklore, but we include a proof for the sake of completeness.

Lemma 3.19. *Let* M *be a multi-graded* R*-module (not necessarily finitely generated) and* $Z \subset X$ *be a closed subset of dimension zero. If* $(\mathrm{Supp}_R(M) \cap X) \subset Z$*, then we have* $H^j_{\mathfrak{N}}(M) = 0$ *for any* $j \geqslant 2$*.*

Proof. Let $i : Z \to X$ be the inclusion of the closed set Z, $\mathcal{M}$ the sheafification $\mathcal{M} = \widetilde{M(\mathbf{n})}$ of M twisted by $\mathbf{n} \in \mathbb{Z}^m$, and $\mathcal{M}\,|_Z$ the restriction of $\mathcal{M}$ to Z. Since the support of $\mathcal{M}$ is contained in Z, then extending $\mathcal{M}\,|_Z$ by zero gives the isomorphism $\mathcal{M} \cong i_*(\mathcal{M}\,|_Z)$ (see [66, Exercise II.1.19(c)]). Using (3.3), [66, Lemma III.2.10] and the Grothendieck vanishing theorem [66, Theorem III.2.7], we get

$$\left[H^j_{\mathfrak{N}}(M)\right]_{\mathbf{n}} \cong H^{j-1}(X,\mathcal{M}) \cong H^{j-1}(X, i_*(\mathcal{M}\,|_Z)) = H^{j-1}(Z, \mathcal{M}\,|_Z) = 0$$

for any $j \geqslant 2$ and $\mathbf{n} \in \mathbb{Z}^m$. ☐

Lemma 3.20. *The following statements hold:*

(i) For each $i \geqslant 0$*,* $\left[H^i_{\mathfrak{N}}(\mathrm{Sym}(I))\right]_{\mathbf{0}}$ *is a finitely generated graded* S*-module.*

(ii) If $\dim(\mathcal{B}) = 0$*, then*

$$\operatorname{rank}\Big(\left[H^1_{\mathfrak{N}}(\mathrm{Sym}(I))\right]_{\mathbf{0}}\Big) = \operatorname{rank}\Big(\left[H^1_{\mathfrak{N}}(\mathcal{R}(I))\right]_{\mathbf{0}}\Big) + \operatorname{rank}\Big(\left[H^1_{\mathfrak{N}}\left(H^0_I(\mathrm{Sym}(I))\right)\right]_{\mathbf{0}}\Big).$$

Proof. (i) The proof of Proposition 3.7(i) applies verbatim.

(ii) From Lemma 1.10, we can make the identification $\mathcal{K} = H^0_I(\mathrm{Sym}(I))$ in the short exact sequence (3.11). Hence, we can obtain the following long exact sequence in local cohomology

$$H^0_{\mathfrak{N}}(\mathcal{R}(I)) \to H^1_{\mathfrak{N}}\big(H^0_I(\mathrm{Sym}(I))\big) \to H^1_{\mathfrak{N}}(\mathrm{Sym}(I)) \to H^1_{\mathfrak{N}}(\mathcal{R}(I)) \to H^2_{\mathfrak{N}}\big(H^0_I(\mathrm{Sym}(I))\big).$$

We clearly have that $H^0_{\mathfrak{N}}(\mathcal{R}(I)) = 0$, and from Lemma 3.19 we get that $H^2_{\mathfrak{N}}\big(H^0_I(\mathrm{Sym}(I))\big) = 0$. Therefore, the assertion follows. □

In the rest of this subsection one of the main tools to be used will be the so-called approximation complexes. These complexes were introduced in [134], and extensively developed in [69], [70] and [71]. In particular, we will consider the $\mathcal{Z}_\bullet$ complex in order to obtain an approximation of a resolution of $\mathrm{Sym}(I)$.

We fix some notations regarding the approximation complexes, and for more details we refer the reader to [71]. Let $K_\bullet = K(f_0, \dots, f_\delta; R)$ be the graded Koszul complex of R-modules:

$$K_\bullet : 0 \to K_{\delta+1} \xrightarrow{d_{\delta+1}} K_\delta \xrightarrow{d_\delta} \dots \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \xrightarrow{d_0} 0$$

associated to the sequence $\{f_0, \dots, f_\delta\}$. For each $i \geqslant 0$, let Z_i be the i-th Koszul cycle and H_i be the i-th Koszul homology, that is $Z_i = \mathrm{Ker}(d_i)$ and $H_i = H_i(K_\bullet)$. Using the Koszul complex $K(y_0, \dots, y_\delta; \mathfrak{A})$, one can construct the approximation complexes $\mathcal{Z}_\bullet$ and $\mathcal{M}_\bullet$ (see [71, Section 4]). The $\mathcal{Z}_\bullet$ complex is given by

$$\mathcal{Z}_\bullet : 0 \to \mathcal{Z}_{\delta+1} \to \mathcal{Z}_\delta \to \cdots \to \mathcal{Z}_1 \to \mathcal{Z}_0 \to 0,$$

where $\mathcal{Z}_i = \big[Z_i \otimes_R \mathfrak{A}\big](i \cdot \mathbf{d}, -i)$ for all $1 \leqslant i \leqslant \delta + 1$. We have that $H_0(\mathcal{Z}_\bullet) \cong \mathrm{Sym}(I)$ and $\mathcal{Z}_{\delta+1} = 0$. Similarly, the $\mathcal{M}_\bullet$ complex is given by

$$\mathcal{M}_\bullet : 0 \to \mathcal{M}_{\delta+1} \to \mathcal{M}_\delta \to \cdots \to \mathcal{M}_1 \to \mathcal{M}_0 \to 0,$$

where $\mathcal{M}_i = \big[H_i \otimes_R \mathfrak{A}\big](i \cdot \mathbf{d}, -i)$ for all $1 \leqslant i \leqslant \delta + 1$.

The next theorem contains the main results of this subsection.

Theorem 3.21. *Let $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow \mathbb{P}^\delta$ be a dominant rational map. If $\dim(\mathcal{B}) = 0$, then the following statements hold:*

(i) $\deg(\mathcal{F}) = \mathrm{rank}\Big(\big[H^0_{\mathfrak{N}}\big(H^1_I(\mathrm{Sym}(I))\big)\big]_{\mathbf{0}}\Big) + 1.$

(ii) We have

$$\deg(\mathcal{F}) \leqslant \mathrm{rank}\Big(\big[H^1_{\mathfrak{N}}\big(\mathrm{Sym}(I)\big)\big]_{\mathbf{0}}\Big) + 1,$$

with equality if I is of linear type.

(iii) In terms of the Koszul cycles Z_i, we get the following upper bound

$$\deg(\mathcal{F}) \leqslant 1 + \sum_{i=0}^{\delta} \dim_{\mathbb{k}} \left(\left[H_{\mathfrak{N}}^{i+1}(Z_i) \right]_{i \cdot \mathbf{d}} \right).$$

Proof. (i) We consider the double complex $F^{\bullet,\bullet} = C_{\mathfrak{N}}^{\bullet} \otimes_R C_I^{\bullet} \otimes_R \mathrm{Sym}(I)$, where $C_{\mathfrak{N}}^{\bullet}$ and $C_I^{\bullet}$ are the Čech complexes corresponding with $\mathfrak{N}$ and I, respectively. We have the spectral sequence

$$E_2^{p,q} = H_{\mathfrak{N}}^p\big(H_I^q(\mathrm{Sym}(I))\big) \Longrightarrow H^{p+q}(\mathrm{Tot}(F^{\bullet,\bullet})) \cong H_{\mathfrak{N}}^{p+q}(\mathrm{Sym}(I)).$$

From Lemma 3.19 we obtain that $E_2^{p,q} = 0$ for $p \geqslant 2$. Therefore, the spectral sequence converges with $E_2^{p,q} = E_\infty^{p,q}$.

The filtration of the term $H^1(\mathrm{Tot}(F^{\bullet,\bullet})) \cong H_{\mathfrak{N}}^1(\mathrm{Sym}(I))$ yields the equality

$$\mathrm{rank}\Big(\big[H_{\mathfrak{N}}^1(\mathrm{Sym}(I))\big]_{\mathbf{0}}\Big) = \mathrm{rank}\Big(\big[H_{\mathfrak{N}}^0\big(H_I^1(\mathrm{Sym}(I))\big)\big]_{\mathbf{0}}\Big) + \mathrm{rank}\Big(\big[H_{\mathfrak{N}}^1\big(H_I^0(\mathrm{Sym}(I))\big)\big]_{\mathbf{0}}\Big),$$

and assembling with Remark 3.18, Corollary 3.12, and Lemma 3.20(ii) we get

$$\deg(\mathcal{F}) = \mathrm{rank}\Big(\big[H_{\mathfrak{N}}^0\big(H_I^1(\mathrm{Sym}(I))\big)\big]_{\mathbf{0}}\Big) + 1.$$

(ii) It follows from Remark 3.18, Corollary 3.12 and Lemma 3.20(ii).

(iii) For any $i \geqslant 0$, we have that $I \cdot H_i = 0$ and so the support of H_i is contained in $\mathcal{B} = V(I)$. Hence, for any $\mathfrak{p} \notin \mathcal{B}$ we have $(\mathcal{M}_\bullet)_{\mathfrak{p}} = 0$. Applying basic properties of approximation complexes (see e.g. [71, Corollary 4.6]), we can obtain that $H_i(\mathcal{Z}_\bullet)_{\mathfrak{p}} = 0$ for any $\mathfrak{p} \notin \mathcal{B}$ and $i \geqslant 1$. Therefore, from Lemma 3.19 we get that $H_{\mathfrak{N}}^j(H_i(\mathcal{Z}_\bullet)) = 0$ for any $j \geqslant 2$ and $i \geqslant 1$.

Let $\{\vartheta_1, \ldots, \vartheta_r\}$ be a set of generators of $\mathfrak{N}$ and $G^{\bullet,\bullet}$ be the corresponding double complex

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{Z}_\delta \otimes_R C_{\mathfrak{N}}^r & \longrightarrow & \mathcal{Z}_{\delta-1} \otimes_R C_{\mathfrak{N}}^r & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z}_0 \otimes_R C_{\mathfrak{N}}^r & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{Z}_\delta \otimes_R C_{\mathfrak{N}}^1 & \longrightarrow & \mathcal{Z}_{\delta-1} \otimes_R C_{\mathfrak{N}}^1 & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z}_0 \otimes_R C_{\mathfrak{N}}^1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{Z}_\delta \otimes_R C_{\mathfrak{N}}^0 & \longrightarrow & \mathcal{Z}_{\delta-1} \otimes_R C_{\mathfrak{N}}^0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z}_0 \otimes_R C_{\mathfrak{N}}^0 & \longrightarrow & 0
\end{array}$$

given by $\mathcal{Z}_\bullet \otimes_R C_{\mathfrak{N}}^\bullet$. The double complex above is written in the second quadrant. Then, the spectral sequence corresponding with the second filtration is given by

$$ {}^{\mathrm{II}}E_2^{p,-q} = \begin{cases} H^p_{\mathfrak{N}}(\mathrm{Sym}(I)) & \text{if } q=0 \\ H^p_{\mathfrak{N}}(H_q(\mathcal{Z}_\bullet)) & \text{if } p \leqslant 1 \text{ and } q \geqslant 1 \\ 0 & \text{otherwise.} \end{cases} $$

Thus, it converges with ${}^{\mathrm{II}}E_2^{p,-q} = {}^{\mathrm{II}}E_\infty^{p,-q}$. In particular, we have $H^1(\mathrm{Tot}(G^{\bullet,\bullet})) \cong H^1_{\mathfrak{N}}(\mathrm{Sym}(I))$.

On the other hand, by computing with the first filtration we get

$$ {}^{\mathrm{I}}E_1^{-p,q} = H^q_{\mathfrak{N}}(\mathcal{Z}_p). $$

Therefore we obtain the following upper bound

$$ \mathrm{rank}\Big(\big[H^1_{\mathfrak{N}}(\mathrm{Sym}(I))\big]_{\mathbf{0}}\Big) \leqslant \sum_{i=0}^{\delta} \mathrm{rank}\Big(\big[H^{i+1}_{\mathfrak{N}}(\mathcal{Z}_i)\big]_{\mathbf{0}}\Big). $$

For each $0 \leqslant i \leqslant \delta$, since $\mathcal{Z}_i = [Z_i \otimes_R \mathfrak{A}]\,(i\cdot\mathbf{d}, -i)$ then we have that

$$ \mathrm{rank}\Big(\big[H^{i+1}_{\mathfrak{N}}(\mathcal{Z}_i)\big]_{\mathbf{0}}\Big) = \mathrm{rank}\Big(\big[H^{i+1}_{\mathfrak{N}}(Z_i)\big]_{i\cdot\mathbf{d}} \otimes_{\Bbbk} S(-i)\Big) = \dim_{\Bbbk}\big(\big[H^{i+1}_{\mathfrak{N}}(Z_i)\big]_{i\cdot\mathbf{d}}\big). $$

Finally, the inequality follows from part (ii). □

Rational maps defined over multi-projective spaces

Here we specialize further our approach to the special case of a multi-graded dominant rational map from a multi-projective space to a projective space. The main results of this subsection are given in Theorem 3.28 and Theorem 3.30, where we provide effective criteria for the birationality of a bi-graded rational map of the form $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ with low bi-degree. We set the following.

Setup 3.22. *Let* $m \geqslant 1$. *For each* $i = 1, \ldots, m$, *let* X_i *be the projective space* $X_i = \mathbb{P}^{r_i}$ *and* A_i *be its coordinate ring* $A_i = \Bbbk[\mathbf{x}_i] = \Bbbk[x_{i,0}, x_{i,1}, \ldots, x_{i,r_i}]$. *Let* $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow \mathbb{P}^\delta$ *be a dominant rational map defined by* $\delta+1$ *multi-homogeneous polynomials* $\mathbf{f} = \{f_0, f_1, \ldots, f_\delta\} \subset R := A_1 \otimes_{\Bbbk} A_2 \otimes_{\Bbbk} \cdots A_m$ *of the same multi-degree* $\mathbf{d} = (d_1, \ldots, d_m)$, *where* $\delta = r_1 + r_2 + \cdots + r_m$ *is the dimension of* X. *Let* I *be the multi-homogeneous ideal generated by* $f_0, f_1, \ldots, f_\delta$. *Let* S *be the homogeneous coordinate ring* $S = \Bbbk[y_0, y_1, \ldots, y_\delta]$ *of* $\mathbb{P}^\delta$. *Let* $\mathfrak{N}$ *be the irrelevant multi-homogeneous ideal of* R, *which is given by* $\mathfrak{N} = \bigoplus_{j_1>0,\ldots,j_m>0} R_{j_1,\ldots,j_m}$.

First we give a description of the local cohomology modules $H^j_{\mathfrak{N}}(R)$, with special attention to its multi-graded structure. We provide a shorter proof than the one obtained in [15, Section 6.1].

Given any subset α of $\{1, \ldots, m\}$, then we define its weight by $\|\alpha\| = \sum_{i\in\alpha} r_i$. For $i \in \{1, \ldots, m\}$, let $\mathfrak{m}_i$ be the maximal irrelevant ideal $\mathfrak{m}_i = (\mathbf{x}_i) = (x_{i,0}, x_{i,1}, \ldots, x_{i,r_i})$. We then have that

$$H^j_{\mathfrak{m}_i}(A_i) \cong \begin{cases} \frac{1}{\mathbf{x}_i}\Bbbk[\mathbf{x}_i^{-1}] & \text{if } j = r_i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.23. *For any* $j \geqslant 0$ *we have that*

$$H^j_{\mathfrak{N}}(R) \;\cong\; \bigoplus_{\substack{\alpha \subset \{1,\ldots,m\} \\ ||\alpha||+1=j}} \left(\bigotimes_{i \in \alpha} \frac{1}{\mathbf{x}_i}\Bbbk[\mathbf{x}_i^{-1}] \right) \otimes_{\Bbbk} \left(\bigotimes_{i \notin \alpha} A_i \right).$$

Proof. First we check that $H^0_{\mathfrak{N}}(R) = H^1_{\mathfrak{N}}(R) = 0$. It is clear that $H^0_{\mathfrak{N}}(R) = 0$, and using (3.2) we get the exact sequence

$$0 \to R \to \bigoplus_{\mathbf{n} \in \mathbb{Z}^m} H^0(X, \mathcal{O}_X(\mathbf{n})) \to H^1_{\mathfrak{N}}(R) \to 0.$$

From the Künneth formula (see [137, Tag 0BEC] for a detailed proof) and [66, Proposition II.5.13] we obtain

$$\begin{aligned} H^0(X, \mathcal{O}_X(\mathbf{n})) &\cong H^0\left(X_1, \mathcal{O}_{X_1}(n_1)\right) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^0\left(X_m, \mathcal{O}_{X_m}(n_m)\right) \\ &\cong [A_1]_{n_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} [A_m]_{n_m} \\ &\cong R_{\mathbf{n}}, \end{aligned}$$

so we conclude that $H^1_{\mathfrak{N}}(R) = 0$.

Let $j \geqslant 2$. Then, the Künneth formula and (3.3) yield the following isomorphisms

$$\begin{aligned} H^j_{\mathfrak{N}}(R) &\cong \bigoplus_{\mathbf{n} \in \mathbb{Z}^m} H^{j-1}(X, \mathcal{O}_X(\mathbf{n})) \\ &\cong \bigoplus_{\mathbf{n} \in \mathbb{Z}^m} \left(\bigoplus_{j_1 + \cdots + j_m = j-1} H^{j_1}\left(X_1, \mathcal{O}_{X_1}(n_1)\right) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^{j_m}\left(X_m, \mathcal{O}_{X_m}(n_m)\right) \right). \end{aligned}$$

For each $i = 1, \ldots, m$ we have that

$$\bigoplus_{n \in \mathbb{Z}} H^{j_i}\left(X_i, \mathcal{O}_{X_i}(n)\right) \cong \begin{cases} A_i & \text{if } j_i = 0 \\ H^{r_i+1}_{\mathfrak{m}_i}(A_i) & \text{if } j_i = r_i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we get the formula

$$H^j_{\mathfrak{N}}(R) \cong \bigoplus_{\substack{j_1+\cdots+j_m=j-1\\ j_i=0 \text{ or } j_i=r_i}} \left(\bigotimes_{\{i|j_i=r_i\}} H^{r_i+1}_{\mathfrak{m}_i}(A_i) \right) \otimes_{\Bbbk} \left(\bigotimes_{\{i|j_i=0\}} A_i \right),$$

which is equivalent to the statement of the proposition. $\square$

Now we give a different proof of Theorem 3.4(iv); in this case we recover the equivalence between the birationality of $\mathcal{F}$ and the equality $\widetilde{\mathfrak{F}_R(I)} = S$. The following result is a generalization of [118, Proposition 1.2].

Proposition 3.24. *Let $\mathcal{F} : \mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \times \cdots \times \mathbb{P}^{r_m} \dashrightarrow \mathbb{P}^{\delta}$ be a dominant rational map with $r_1 + r_2 + \ldots + r_m = \delta$. Then, the map $\mathcal{F}$ is birational if and only if for all $n \geqslant 1$ we have*

$$[I^n]_{n\cdot\mathbf{d}} = \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}}.$$

Proof. From Theorem 3.4(iii), the equality above implies the birationality of $\mathcal{F}$.

For the other implication, let us assume that $\mathcal{F}$ is birational. Since $\mathcal{F}$ is dominant, then $S = \Bbbk[y_0, \ldots, y_\delta]$ is isomorphic to the coordinate ring $S \cong \Bbbk[I_{\mathbf{d}}] = \Bbbk[f_0, \ldots, f_\delta]$ of the image. Let T be the multi-Veronese subring $T = \Bbbk[R_{\mathbf{d}}]$, then after regrading we have a canonical inclusion $S \cong \Bbbk[I_{\mathbf{d}}] \subset T$ of standard graded $\Bbbk$-algebras. From Lemma 3.5 and the assumption of birationality we get

$$[T : S] = \deg(\mathcal{F}) = 1.$$

So we have $\mathrm{Quot}(S) = \mathrm{Quot}(T)$ and the following canonical inclusions

$$S \subset T \subset \mathrm{Quot}(T) = \mathrm{Quot}(S).$$

Let $n \geqslant 1$. It is enough to prove that for any $w \in \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}} \subset T_n$, we have that w is integral over S. Indeed, since S is integrally closed, it will imply that $w \in S_n \cong [I^n]_{n\cdot\mathbf{d}}$.

Let $w \in \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}}$. We shall prove the equivalent condition that $S[w]$ is a finitely generated S-module (see e.g. [7, Proposition 5.1]). From the condition $w \in \left[(I^n)^{\mathrm{sat}}\right]_{n\cdot\mathbf{d}}$, we can choose some $r > 0$ such that

$$T_{rn} \cdot w = R_{rn\cdot\mathbf{d}} \cdot w \subset [I^n]_{(r+1)n\cdot\mathbf{d}}.$$

We claim that for any $q \geqslant r+1$ we have $w^q \in S \cdot R_{rn\cdot\mathbf{d}}$. If we prove this claim, then it will follow that $S[w]$ is a finitely generated S-module.

Let $\{F_1, \ldots, F_c\}$ be a minimal generating set of the ideal I^n. For $q = r+1$, since $w^r \in R_{rn\cdot\mathbf{d}}$ we can write

$$w^{r+1} = w^r w = h_1F_1 + h_1F_1 + \cdots + h_cF_c,$$

where $\deg(h_i) = rn \cdot \mathbf{d}$ for each $i = 1, \dots, c$. For $q = r+2$, since $h_i \in R_{rn\cdot\mathbf{d}}$ we get

$$w^{r+2} = ww^{r+1} = \sum_{i=1}^{c} (h_i w)\, F_i = \sum_{i=1}^{c} \left(\sum_{j=1}^{c} h_{ij} F_j \right) F_i = \sum_{i,j} h_{ij} F_i F_j,$$

where each h_{ij} has degree $\deg(h_{ij}) = rn \cdot \mathbf{d}$. Following this inductive process, we have that for each $q \geqslant r+1$ we can write

$$w^q = \sum_{\beta} h_\beta \mathbf{F}^\beta,$$

where $\deg(h_\beta) = rn \cdot \mathbf{d}$ for each multi-index β. This gives us the claim that $w^q \in S \cdot R_{rn\cdot\mathbf{d}}$ for each $q \geqslant r+1$. □

From Proposition 3.24 we deduce that for single-graded birational maps with non saturated base ideal, the module I^{sat}/I is generated by elements of degree $\geqslant d+1$.

Corollary 3.25. *Let $\mathcal{F} : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ be a birational map whose base ideal $I = (f_0, \cdots, f_r)$ is given by $r+1$ relatively prime polynomials of the same degree d. Then, we have that*

$$\left[I^{\text{sat}}/I\right]_{\leqslant d} = 0.$$

Proof. From Proposition 3.24 we already have $[I^{\text{sat}}]_d = I_d$. If we assume that there exists $0 \neq h \in [I^{\text{sat}}]_{d-1}$, then we get the contradiction $I_d = (x_0 h, x_1 h, \cdots, x_r h)$. Therefore, we have $[I^{\text{sat}}/I]_{\leqslant d} = 0$. □

For multi-graded birational maps the previous condition must not be necessarily satisfied.

Example 3.26. *Let $\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ be the birational map given by*

$$(x_{1,0} : x_{1,1}) \times (x_{2,0} : x_{2,1}) \mapsto (x_{1,0}x_{2,0} : x_{1,1}x_{2,0} : x_{1,1}x_{2,1}).$$

Here, the base ideal $I = (x_{1,0}x_{2,0},\, x_{1,1}x_{2,0},\, x_{1,1}x_{2,1})$ is generated by forms of bi-degree $(1,1)$ and $\mathfrak{N} = (x_{1,0}, x_{1,1}) \cap (x_{2,0}, x_{2,1})$. The map $\mathcal{F}$ is birational, but we have that $I^{\text{sat}} = (I : \mathfrak{N}^\infty) = (x_{1,1}, x_{2,0})$ and so

$$\left[I^{\text{sat}}/I\right]_{(1,0)} \neq 0 \qquad \textit{and} \qquad \left[I^{\text{sat}}/I\right]_{(0,1)} \neq 0.$$

From now on, we focus on a dominant rational map of the form $\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. We shall adapt our previous results to this case and obtain a general upper bound for the degree of $\mathcal{F}$. More interestingly, we give a criterion for birationality when the bi-degrees of the f_i's are of the form $\mathbf{d} = (d_1, d_2)$ and $d_1 = 1$. This result extends the work of [16], where a criterion was given for the bi-degrees $(1,1)$ and $(1,2)$. Also, in the case $\mathbf{d} = (d_1, d_2) = (2,2)$ we provide a general characterization for the birationality of $\mathcal{F}$ (see [16, Theorem 16] for a more specific result).

Proposition 3.27. *Let* $\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ *be a dominant rational map such that* $\dim(\mathcal{B}) = 0$. *Then, we have the inequality*

$$\deg(\mathcal{F}) \leqslant 1 + (d_1 - 1)(d_2 - 1) + \dim_{\mathbb{k}}\left(\left[\mathrm{I}^{\mathrm{sat}}/\mathrm{I}\right]_{\mathbf{d}}\right).$$

Proof. From Theorem 3.21(iii) we have the inequality

$$\deg(\mathcal{F}) \leqslant 1 + \dim_{\mathbb{k}}\left(\left[\mathrm{H}^3_{\mathfrak{N}}(\mathrm{Z}_2)\right]_{2\cdot\mathbf{d}}\right) + \dim_{\mathbb{k}}\left(\left[\mathrm{H}^2_{\mathfrak{N}}(\mathrm{Z}_1)\right]_{\mathbf{d}}\right) + \dim_{\mathbb{k}}\left(\left[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{Z}_0)\right]_{\mathbf{d}}\right).$$

By Proposition 3.23 and the fact that $\mathrm{Z}_0 \cong \mathrm{R}$ and $\mathrm{Z}_2 \cong \mathrm{R}(-3 \cdot \mathbf{d})$, we obtain the isomorphisms $\mathrm{H}^1_{\mathfrak{N}}(\mathrm{Z}_0) = 0$ and

$$\mathrm{H}^3_{\mathfrak{N}}(\mathrm{Z}_2) \cong \mathrm{H}^3_{\mathfrak{N}}(\mathrm{R})(-3 \cdot \mathbf{d}) \cong \left(\frac{1}{\mathbf{x}_1}\mathbb{k}[\mathbf{x}_1^{-1}]\right)(-3d_1) \otimes_{\mathbb{k}} \left(\frac{1}{\mathbf{x}_2}\mathbb{k}[\mathbf{x}_2^{-1}]\right)(-3d_2).$$

Thus, we get that

$$\dim_{\mathbb{k}}\left(\left[\mathrm{H}^3_{\mathfrak{N}}(\mathrm{Z}_2)\right]_{2\cdot\mathbf{d}}\right) = \dim_{\mathbb{k}}\left(\left[\frac{1}{\mathbf{x}_1}\mathbb{k}[\mathbf{x}_1^{-1}]\right]_{-d_1} \otimes_{\mathbb{k}} \left[\frac{1}{\mathbf{x}_2}\mathbb{k}[\mathbf{x}_2^{-1}]\right]_{-d_2}\right) = (d_1 - 1)(d_2 - 1).$$

The exact sequences

$$0 \to \mathrm{Z}_1 \to \mathrm{R}^3(-d_1, -d_2) \to \mathrm{I} \to 0$$
$$0 \to \mathrm{I} \to \mathrm{R} \to \mathrm{R}/\mathrm{I} \to 0$$

and Proposition 3.23 yield the isomorphisms

$$\dim_{\mathbb{k}}\left(\left[\mathrm{H}^2_{\mathfrak{N}}(\mathrm{Z}_1)\right]_{\mathbf{d}}\right) = \dim_{\mathbb{k}}\left(\left[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{I})\right]_{\mathbf{d}}\right) = \dim_{\mathbb{k}}\left(\left[\mathrm{H}^0_{\mathfrak{N}}(\mathrm{R}/\mathrm{I})\right]_{\mathbf{d}}\right) = \dim_{\mathbb{k}}\left(\left[\mathrm{I}^{\mathrm{sat}}/\mathrm{I}\right]_{\mathbf{d}}\right).$$

Therefore, by combining these computations, we get the claimed upper bound. □

Theorem 3.28. *Let* $\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ *be a dominant rational map such that* $\dim(\mathcal{B}) = 0$ *and* $\mathbf{d} = (1, d_2)$. *Then,* $\mathcal{F}$ *is birational if and only if* $\mathrm{I}_{\mathbf{d}} = [\mathrm{I}^{\mathrm{sat}}]_{\mathbf{d}}$.

Proof. We get one implication from Proposition 3.24 and the other by specializing the data in the inequality of Proposition 3.27. □

To illustrate this theorem, let $\mathcal{F}$ be as above and assume moreover that there exists a nonzero syzygy of I of bi-degree $(0, 1)$. As in [16, Remark 10], we get that $x_{2,0}(\sum_{i=0}^{2} \alpha_i f_i) - x_{2,1}(\sum_{i=0}^{2} \beta_i f_i) = 0$ for some α_i's and β_i's in $\mathbb{k}$ and hence we deduce that there exist three polynomials p, q, r of bi-degree $(1, d_2 - 1)$ such that $\mathrm{I} = (x_{2,0}p,\ x_{2,1}p,\ x_{2,0}q + x_{2,1}r)$. Therefore, the ideal I admits a

Hilbert-Burch presentation of the form

$$F_\bullet : \quad 0 \to R(-1,-1-d_2) \oplus R(-2,-2d_2+1) \to R(-1,-d_2)^3 \to R.$$

Studying the two spectral sequences coming from the double complex $F_\bullet \otimes_R C^\bullet_{\mathfrak{N}}$, together with Proposition 3.23, it is then easy to see that $[I^{\text{sat}}/I]_{\mathbf{d}} \cong \left[H^0_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}} = 0$. Thus, Theorem 3.28 implies that $\mathcal{F}$ is birational, a fact that can be deduced more directly and that is the main ingredient to ensure birationality in [127]. But Theorem 3.28 provides actually a finer result. Indeed, suppose that the ideal I admits the following more general Hilbert-Burch presentation

$$0 \to R(-1,-\mu-d_2) \oplus R(-2,-2d_2+\mu) \to R(-1,-d_2)^3 \to R$$

where μ is a positive integer. Then, a similar computation shows that $\left[H^0_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}} \cong \left[H^2_{\mathfrak{N}}(R)\right]_{(0,-\mu)}$ and from here we deduce that $\mathcal{F}$ cannot be a birational map if $\mu > 1$.

Lemma 3.29. *Let* $\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ *be a dominant rational map such that* $\dim(\mathcal{B}) = 0$ *and* $\mathbf{d} = (2,2)$. *Then,* $I_{\mathbf{d}} = [I^{\text{sat}}]_{\mathbf{d}}$ *if and only if* $\deg(\mathcal{B}) = 6$.

Proof. From (3.2) we have the short exact sequence

$$0 \to \left[H^0_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}} \to [R/I]_{\mathbf{d}} \to H^0\left(X, (\mathcal{O}_X/I^\sim)(\mathbf{d})\right) \to \left[H^1_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}} \to 0.$$

Using [16, Lemma 5] we deduce that $\left[H^1_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}} = 0$. Therefore, we obtain

$$\begin{aligned} \deg(\mathcal{B}) &= \dim_{\mathbb{k}}\left(H^0\left(X, (\mathcal{O}_X/I^\sim)(\mathbf{d})\right)\right) \\ &= \dim_{\mathbb{k}}\left([R/I]_{\mathbf{d}}\right) - \dim_{\mathbb{k}}\left(\left[H^0_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}}\right) = 6 - \dim_{\mathbb{k}}\left(\left[H^0_{\mathfrak{N}}(R/I)\right]_{\mathbf{d}}\right) \end{aligned}$$

from the exact sequence above, and so the claimed result follows. □

Theorem 3.30. *Let* $\mathcal{F} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ *be a dominant rational map. Suppose that* $\dim(\mathcal{B}) = 0$ *and* $\mathbf{d} = (2,2)$. *Then,* $\mathcal{F}$ *is birational if and only if the following conditions are satisfied:*

(i) $I_{\mathbf{d}} = [I^{\text{sat}}]_{\mathbf{d}}$.

(ii) I *is not locally a complete intersection at its minimal primes.*

Proof. The degree formula of Theorem 3.16 applied to our setting gives

$$\deg(\mathcal{F}) = 8 - e(\mathcal{B}).$$

Hence, we deduce that $e(\mathcal{B}) \leqslant 7$ and that $\mathcal{F}$ is birational if and only if $e(\mathcal{B}) = 7$. We know that $\deg(\mathcal{B}) \leqslant e(\mathcal{B})$, and that $\deg(\mathcal{B}) = e(\mathcal{B})$ if and only if I is locally a complete intersection at its minimal primes. Moreover, we have already seen that the condition $I_{\mathbf{d}} = [I^{\text{sat}}]_{\mathbf{d}}$ is necessary for the birationality of $\mathcal{F}$ (Proposition 3.24) and that it is equivalent to $\deg(\mathcal{B}) = 6$ (Lemma 3.29).

Therefore, assuming $I_{\mathbf{d}} = [I^{\text{sat}}]_{\mathbf{d}}$, we have that I is not locally a complete intersection at its minimal primes if and only if

$$6 = \deg(\mathcal{B}) < e(\mathcal{B}) = 7,$$

and the later one is equivalent to the birationality of $\mathcal{F}$. ☐

An explicit upper bound for the degree of a rational map defined over a projective space

In this subsection we consider the more specific case of single-graded dominant rational maps. The main result here is Theorem 3.34 where the upper bound for the degree of a rational map given in Theorem 3.21(iii), is expressed solely in terms of the Hilbert functions of R/I and I^{sat}/I, instead of some local cohomology modules of Koszul cycles. We also show that this upper bound is sharp in some cases. We set the following.

Setup 3.31. *Let* R *be the standard graded polynomial ring* $R = \mathbb{k}[x_0, x_1, \ldots, x_r]$*, and* $\mathfrak{m}$ *be the maximal irrelevant ideal* $\mathfrak{m} = (x_0, \ldots, x_r)$*. Let* $\mathcal{F} : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ *be a dominant rational map defined by* $r+1$ *homogeneous polynomials* $\mathbf{f} = \{f_0, f_1, \ldots, f_r\} \subset R$ *of the same degree* d*. Let* I *be the homogeneous ideal generated by* $f_0, f_1, \ldots, f_r$*. Let* S *be the standard graded polynomial ring* $S = \mathbb{k}[y_0, y_1, \ldots, y_r]$*. Let* $\mathfrak{A}$ *be the bigraded polynomial ring* $\mathfrak{A} = R \otimes_{\mathbb{k}} S$*, where* $\text{bideg}(x_i) = (1, 0)$ *and* $\text{bideg}(y_j) = (0, 1)$*. For any graded* R*-module* M*, we set* $M^{\vee} = {}^*\text{Hom}_{\mathbb{k}}(M, \mathbb{k})$ *to be the graded Matlis dual of* M *(see e.g.* [19, Section 3.6]*).*

The following lemma is equivalent to [21, Lemma 1] in our setting; we include a proof for the sake of completeness and the convenience of the reader.

Lemma 3.32. *Let* Z_i *and* H_i *be the cycles and homology modules of the Koszul complex* $K(\mathbf{f}; R)$*, respectively. Assume that* $\dim(R/I) \leqslant 1$ *and let* $\xi = (r+1)(d-1)$*. Then,*

(i) $Z_{r+1} = 0$*,* $Z_r \cong R(-(r+1)d)$*,* $Z_0 = R$*,* $H_i = 0$ *for* $i > 1$*,* $H_1 = 0$ *if and only if* $\dim(R/I) = 0$*. If* $\dim(R/I) = 1$*, then* $H_1 \cong \omega_{R/I}(-\xi)$*.*

(ii) If $r \geqslant 2$ *and* $1 \leqslant p < r$*, then*

$$H_{\mathfrak{m}}^q(Z_p) \cong \begin{cases} H_{\mathfrak{m}}^{q-2}(R/I) & \text{if } p = 1 \text{ and } q \leqslant r \\ H_{q-p-1}^{\vee}(-\xi) & \text{if } 2 \leqslant p < r \text{ and } q \leqslant r \\ Z_{r-p}^{\vee}(-\xi) & \text{if } q = r+1. \end{cases}$$

Proof. (i) This part follows from well known properties of the Koszul complex (see e.g. [19, Section 1.6]).

(ii) We only need to compute the local cohomology modules of Z_p for $1 \leqslant p < r$.

Let $2 \leqslant \ell < r$. We denote by $K_\bullet^{>\ell}$ the truncated Koszul complex

$$K_\bullet^{>\ell} : \quad 0 \to K_{r+1} \to K_r \to \cdots \to K_{\ell+1} \to Z_\ell \to 0,$$

which is exact from the condition $\dim(R/I) \leqslant 1$. Let $F^{\bullet,\bullet}$ be the double complex given by $F^{\bullet,\bullet} = K_\bullet^{>\ell} \otimes_R C_m^\bullet$. The exactness of $K_\bullet^{>\ell}$ implies that $H^\bullet\left(\mathrm{Tot}(F^{\bullet,\bullet})\right) = 0$. Hence computing with the first filtration we get the spectral sequence ${}^{I}E_1^{-p,q} = H_m^q(K_p^{>\ell}) \Rightarrow 0$, which at the first page is given by

$$\begin{array}{ccccccccc}
H_m^{r+1}(K_{r+1}) & \longrightarrow & H_m^{r+1}(K_r) & \longrightarrow & \cdots & \longrightarrow & H_m^{r+1}(K_{\ell+1}) & \longrightarrow & H_m^{r+1}(Z_\ell) \\
0 & & 0 & & \cdots & & 0 & & H_m^{r}(Z_\ell) \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
0 & & 0 & & \cdots & & 0 & & H_m^{0}(Z_\ell).
\end{array}$$

From the graded local duality theorem (see e.g. [19, Theorem 3.6.19]) and the self-duality of the Koszul complex, we have the following isomorphisms of complexes

$$H_m^{r+1}(K_\bullet) \cong \left(\mathrm{Hom}_R\left(K_\bullet, R(-r-1)\right)\right)^\vee \cong \left(K_\bullet[r+1]\left((r+1)d-r-1\right)\right)^\vee \cong \left(K_\bullet[r+1]\right)^\vee(-\xi),$$

where $[r+1]$ denotes homological shift degree. So the top row of the diagram above is given by the complex

$$K_0^\vee(-\xi) \to K_1^\vee(-\xi) \to \cdots \to K_{r+1-(\ell+1)}^\vee(-\xi) \to H_m^{r+1}(Z_\ell).$$

For each $q \leqslant r$, when we compute cohomology in the page $r+3-q$, we get the exact sequence

$$0 \to {}^{I}E_{r+3-q}^{-(\ell+r+2-q),r+1} \to H_{q-\ell-1}^\vee(-\xi) \to H_m^q(Z_\ell) \to {}^{I}E_{r+3-q}^{-\ell,q} \to 0.$$

Since ${}^{I}E_{r+3-q}^{-(\ell+r+2-q),r+1} = {}^{I}E_\infty^{-(\ell+r+2-q),r+1} = 0$ and ${}^{I}E_{r+3-q}^{-\ell,q} = {}^{I}E_\infty^{-\ell,q} = 0$, we get the isomorphism $H_m^q(Z_\ell) \cong H_{q-\ell-1}^\vee(-\xi)$ when $q \leqslant r$.

In the case of $q = r+1$, we have the exact sequence

$$K_{r+1-(\ell+2)}^\vee(-\xi) \to K_{r+1-(\ell+1)}^\vee(-\xi) \to H_m^{r+1}(Z_\ell) \to 0,$$

that induces the isomorphism $H_m^{r+1}(Z_\ell) \cong Z_{r-\ell}^\vee(-\xi)$.

When $\ell = 1$, we consider the truncated Koszul complex

$$K_\bullet^{>1}: \quad 0 \to K_{r+1} \to K_r \to \cdots \to K_2 \to Z_1 \to 0,$$

that is not exact only at the module Z_1. The double complex $G^{\bullet,\bullet} = K_\bullet^{>1} \otimes_R C_m^\bullet$ now yields the spectral sequence

$${}^{I}E_1^{-p,q} = H_m^q(K_p^{>\ell}) \Longrightarrow H^{-p+q}\left(G^{\bullet,\bullet}\right) = \begin{cases} H_m^1(H_1) & \text{if } -p+q=0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus again we have the exact sequence

$$K^{\vee}_{r-2}(-\xi) \to K^{\vee}_{r-1}(-\xi) \to H^{r+1}_{\mathfrak{m}}(Z_1) \to 0,$$

and this gives us the isomorphism $H^{r+1}_{\mathfrak{m}}(Z_1) \cong Z^{\vee}_{r-1}(-\xi)$.

Finally, using the following two short exact sequences

$$0 \to Z_1 \to K_1 \to I \to 0$$
$$0 \to I \to R \to R/I \to 0$$

we can obtain the isomorphisms $H^{q}_{\mathfrak{m}}(Z_1) \cong H^{q-1}_{\mathfrak{m}}(I) \cong H^{q-2}_{\mathfrak{m}}(R/I)$ for $q \leqslant r$. □

Since the linear type condition has almost no geometrical meaning, we briefly restate the equality of Theorem 3.21(ii) in the locally complete intersection case.

Lemma 3.33. *Let $\mathcal{F} : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ be a dominant rational map with a dimension 1 base ideal I. If I is locally a complete intersection at its minimal primes then*

$$\deg(\mathcal{F}) = \operatorname{rank}\Big(\big[H^1_{\mathfrak{m}}(\operatorname{Sym}(I))\big]_0\Big) + 1.$$

Proof. From either [71, Section 5] or [134, Proposition 3.7] we get that I is of linear type. Thus, the assertion follows from Theorem 3.21(ii). □

The next theorem translates Theorem 3.21(iii) in terms of the Hilbert functions of R/I and I^{sat}/I.

Theorem 3.34. *Let $\mathcal{F} : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ be a dominant rational map with base ideal I. If $\dim(R/I) \leqslant 1$, then we have the following upper bound*

$$\deg(\mathcal{F}) \leqslant 1 + \binom{d-1}{r} + \dim_{\mathbb{k}}\left(\left[I^{\text{sat}}/I\right]_d\right) + \sum_{i=2}^{r-1} \dim_{\mathbb{k}}\Big([R/I]_{(r+1-i)d-r-1}\Big).$$

Proof. Since $Z_0 = R$ and $Z_r \cong R(-(r+1)d)$, we have $H^1_{\mathfrak{m}}(Z_0) = 0$ and

$$\dim_{\mathbb{k}}\left(\left[H^{r+1}_{\mathfrak{m}}(Z_r)\right]_{rd}\right) = \dim_{\mathbb{k}}\left(\left[\frac{1}{\mathbf{x}}\mathbb{k}[\mathbf{x}^{-1}]\right]_{-d}\right) = \binom{(d-r-1)+r}{r} = \binom{d-1}{r}.$$

From Lemma 3.32 we obtain that

$$\dim_{\mathbb{k}}\left(\left[H^2_{\mathfrak{m}}(Z_1)\right]_d\right) = \dim_{\mathbb{k}}\left(\left[H^0_{\mathfrak{m}}(R/I)\right]_d\right) = \dim_{\mathbb{k}}\left(\left[I^{\text{sat}}/I\right]_d\right)$$

and

$$\dim_{\Bbbk}\left(\left[H_{\mathfrak{m}}^{i+1}(Z_i)\right]_{id}\right) = \dim_{\Bbbk}\left(\left[H_0^{\vee}\right]_{(i-r-1)d+r+1}\right) = \dim_{\Bbbk}\left([R/I]_{(r+1-i)d-r-1}\right)$$

for any $2 \leqslant i \leqslant r-1$. Finally, by substituting these computations in Theorem 3.21(iii), we obtain the required upper bound. $\square$

To end this subsection, we show that the above upper bound becomes sharp for dominant plane rational maps when the base ideal is of linear type and is defined by polynomials degree $d \leqslant 3$.

Proposition 3.35. *Let $\mathcal{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a dominant rational map with a dimension 1 base ideal I. Then, the following statements hold:*

(i) $\deg(\mathcal{F}) \leqslant \frac{(d-1)(d-2)}{2} + \dim_{\Bbbk}\left([I^{\text{sat}}/I]_d\right) + 1.$

(ii) If I is of linear type and is generated in degree $d \leqslant 3$, then

$$\deg(\mathcal{F}) = \frac{(d-1)(d-2)}{2} + \dim_{\Bbbk}\left(\left[I^{\text{sat}}/I\right]_d\right) + 1.$$

Proof. (i) It follows from Theorem 3.34.

(ii) From Theorem 3.21(ii), the linear type assumption implies $\deg(\mathcal{F}) = \operatorname{rank}\left(\left[H_{\mathfrak{m}}^1(\operatorname{Sym}(I))\right]_0\right) + 1$. The spectral sequence ${}^{I}E_1^{-p,q} = H_{\mathfrak{m}}^q(\mathcal{Z}_p)$ of the proof of Theorem 3.21(iii) is given by:

$$\begin{array}{ccccc}
H_{\mathfrak{m}}^3(\mathcal{Z}_2) & \longrightarrow & H_{\mathfrak{m}}^3(\mathcal{Z}_1) & \longrightarrow & H_{\mathfrak{m}}^3(\mathcal{Z}_0) \\
0 & & H_{\mathfrak{m}}^2(\mathcal{Z}_1) & & 0 \\
0 & & 0 & & 0 \\
0 & & 0 & & 0
\end{array}$$

Therefore, if we prove that $\left[H_{\mathfrak{m}}^3(\mathcal{Z}_1)\right]_0 = 0$ then the convergence of this spectral sequence implies the required equality. Since $\mathcal{Z}_1 = \left[Z_1 \otimes_R \mathfrak{A}\right](d,-1)$, then it is enough to check that $\left[H_{\mathfrak{m}}^3(Z_1)\right]_d = 0$. The short exact sequence

$$0 \to Z_1 \to R^3(-d) \to I \to 0$$

yields the following exact sequence

$$0 \to H_{\mathfrak{m}}^2(I) \to H_{\mathfrak{m}}^3(Z_1) \to H_{\mathfrak{m}}^3(R^3)(-d),$$

and so we have $\left[H^3_{\mathfrak{m}}(Z_1)\right]_d \cong \left[H^2_{\mathfrak{m}}(I)\right]_d \cong \left[H^1_{\mathfrak{m}}(R/I)\right]_d$. Finally, from [67, Theorem 1.2(ii)] we have that $\operatorname{end}(H^1_{\mathfrak{m}}(R/I)) \leqslant 2d-4$, and so under the assumption $d \leqslant 3$ we have $\left[H^1_{\mathfrak{m}}(R/I)\right]_d = 0$. □

3.3 Multi-graded Jacobian dual criterion of birationality

A rational map is birational if and only if its degree is equal to one, so the results we have previously developed provide birationality criteria. Nevertheless, because of its theoretical and practical importance, some more specific techniques have been developed to decide birationality, mostly for single-graded rational maps. In particular, it has been shown that birationality is controlled by a single numerical invariant that corresponds to the rank of a certain matrix called the *Jacobian dual matrix* (see [130], [129], [46, §2.3 and §2.4] and [16, Section 2.2]). In this section, we extend this theory to the multi-graded setting. In §4.3.1, the multi-graded version of the Jacobian dual matrix is introduced and a general birationality criterion is proved (Theorem 3.39). As an illustration, a very simple birationality criterion is deduced for certain monomial multi-graded maps (Corollary 3.41). Then, in §4.3.2, we investigate how birationality can be detected by using only the syzygies of the base ideal I of a rational map, instead of the whole collection of equations of the Rees algebra of I (Proposition 3.43), which are required for the Jacobian dual matrix. Under the assumption that I is of linear type, we also obtain a syzygy-based birationality criterion (Theorem 3.44).

In this section we use the same notations and conventions of §4.1.1. If the dominant rational map $\mathcal{F} : X = X_1 \times X_2 \times \cdots \times X_m \dashrightarrow Y$ has an inverse, then it is denoted by

$$\mathcal{G} : Y \dashrightarrow (X_1, X_2, \ldots, X_m).$$

Each rational map $Y \dashrightarrow X_i \subset \mathbb{P}^{r_i}$ is defined by $r_i + 1$ homogeneous polynomials $\mathbf{g}_i = \{g_{i,0}, g_{i,1}, \ldots, g_{i,r_i}\} \subset S$ of the same degree. For each $i = 1, \ldots, m$, we set J_i to be the homogeneous ideal generated by $\mathbf{g}_i$.

Jacobian dual matrices and the main criterion

We begin this section with the following preliminary lemma which is based on [16, Lemma 1], [129, Proposition 2.1] and [46, Theorem 2.18].

Lemma 3.36. *Assume that $\mathcal{F}$ is a birational map with inverse $\mathcal{G}$. Let $I = (\mathbf{f})$ and $J_1 = (\mathbf{g}_1), \ldots, J_m = (\mathbf{g}_m)$. Then, the identity map of $\mathbb{k}[\mathbf{x}, \mathbf{y}]$ induces a $\mathbb{k}$-algebra isomorphism between the Rees algebra $\mathcal{R}_R(I)$ and the multi-graded Rees algebra $\mathcal{R}_S(J_1 \oplus J_2 \oplus \cdots \oplus J_m)$.*

Proof. First we note that both algebras can be identified as a quotient of $R \otimes_{\mathbb{k}} S \cong \frac{\mathbb{k}[\mathbf{x},\mathbf{y}]}{(\mathfrak{a}_1,\ldots,\mathfrak{a}_m,\mathfrak{b})}$. The algebra $\mathcal{R}_R(I)$ has a presentation given by

$$\begin{array}{rcl} \frac{\Bbbk[\mathbf{x}]}{(\mathfrak{a}_1,\ldots,\mathfrak{a}_m)}[\mathbf{y}] & \twoheadrightarrow & \mathcal{R}_R(I)=R[\mathbf{f}t] \\ y_i & \mapsto & f_it. \end{array}$$

Let $\overline{\mathcal{I}}=(\mathcal{I},\mathfrak{a}_1,\ldots,\mathfrak{a}_m)/(\mathfrak{a}_1,\ldots,\mathfrak{a}_m)$ denote the kernel of this map. Since Y can be identified with $\mathrm{Proj}(\Bbbk[\mathbf{f}])$ and the two algebras $\Bbbk[\mathbf{f}]$ and $\Bbbk[\mathbf{f}t]$ are isomorphic, then we get $\mathfrak{b}=\mathrm{Ker}(\Bbbk[\mathbf{y}]\twoheadrightarrow\Bbbk[\mathbf{f}])=\mathrm{Ker}(\Bbbk[\mathbf{y}]\twoheadrightarrow\Bbbk[\mathbf{f}t])\subset\mathcal{I}$, as required.

Similarly, the algebra $\mathcal{R}_S(J_1\oplus\cdots\oplus J_m)$ has a presentation

$$\begin{array}{rcl} \frac{\Bbbk[\mathbf{y}]}{\mathfrak{b}}[\mathbf{x}] & \twoheadrightarrow & \mathcal{R}_S(J_1\oplus\cdots\oplus J_m)=S[\mathbf{g}_1t_1,\ldots,\mathbf{g}_mt_m] \\ x_{i,j} & \mapsto & g_{i,j}t_i. \end{array}$$

We denote by $\overline{\mathcal{J}}=(\mathcal{J},\mathfrak{b})/\mathfrak{b}$ the kernel of this map. For each $i=1,\ldots,m$, we can identify X_i with $\mathrm{Proj}(\Bbbk[\mathbf{g}_i])$ and as before we get $\mathfrak{a}_i=\mathrm{Ker}(\Bbbk[\mathbf{x}_i]\twoheadrightarrow\Bbbk[\mathbf{g}_i])=\mathrm{Ker}(\Bbbk[\mathbf{x}_i]\twoheadrightarrow\Bbbk[\mathbf{g}_it_i])\subset\mathcal{J}$.

Since now we can regard $\mathcal{R}_R(I)$ and $\mathcal{R}_S(J_1\oplus\cdots\oplus J_m)$ as quotients of $\frac{\Bbbk[\mathbf{x},\mathbf{y}]}{(\mathfrak{a}_1,\ldots,\mathfrak{a}_m,\mathfrak{b})}$, then it is enough to prove that $\mathcal{J}\subset(\mathcal{I},\mathfrak{a}_1,\ldots,\mathfrak{a}_m)$ and that $\mathcal{I}\subset(\mathcal{J},\mathfrak{b})$.

Let $F(\mathbf{y},\mathbf{x}_1,\ldots,\mathbf{x}_m)\in\mathcal{J}$ be multi-homogeneous, then we have

$$F(\mathbf{y},\mathbf{g}_1t_1,\ldots,\mathbf{g}_mt_m)=0\in S[\mathbf{g}_1t_1,\ldots,\mathbf{g}_mt_m]$$

and using the multi-homogeneity of F we get $F(\mathbf{y},\mathbf{g}_1,\ldots,\mathbf{g}_m)=0\in S$. From the canonical injection $S\cong\Bbbk[\mathbf{f}]\hookrightarrow R$ we make the substitution $y_i\mapsto f_i$, and we obtain

$$F(\mathbf{f},\mathbf{g}_1(\mathbf{f}),\ldots,\mathbf{g}_m(\mathbf{f}))=0\in R.$$

By the assumption of $\mathcal{F}$ being birational, there exist nonzero multi-homogeneous forms $D_1,\ldots,D_m$ in R, possibly of different multi-degrees, such that

$$\mathbf{g}_1(\mathbf{f})=D_1\mathbf{x}_1,\ \mathbf{g}_2(\mathbf{f})=D_2\mathbf{x}_2,\ \ldots,\ \mathbf{g}_m(\mathbf{f})=D_m\mathbf{x}_m.$$

Again, from the multi-homogeneity of F we get

$$F(\mathbf{f},\mathbf{g}_1(\mathbf{f}),\ldots,\mathbf{g}_m(\mathbf{f}))=D_1^{\alpha_1}\cdots D_m^{\alpha_m}F(\mathbf{f},\mathbf{x}_1,\ldots,\mathbf{x}_m)=0\in R,$$

and so $F(\mathbf{f},\mathbf{x}_1,\ldots,\mathbf{x}_m)=0$ because R is an integral domain. From the identification $\Bbbk[\mathbf{f}]\cong\Bbbk[\mathbf{f}t]$ we get

$$F(\mathbf{f}t,\mathbf{x}_1,\ldots,\mathbf{x}_m)=0\in R[\mathbf{f}t],$$

then by definition we get $F\in(\mathcal{I},\mathfrak{a}_1,\ldots,\mathfrak{a}_m)$. Therefore, $\mathcal{J}\subset(\mathcal{I},\mathfrak{a}_1,\ldots,\mathfrak{a}_m)$.

We can prove the other containment with similar arguments. □

Let $(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathcal{J}) \subset \mathbb{k}[\mathbf{x}, \mathbf{y}]$ be the defining equations of the Rees algebra $\mathcal{R}_R(I)$. We shall adopt the following notation.

Notation 3.37. *For each $1 \leqslant i \leqslant m$, let $\{h_{i,1}, \ldots, h_{i,k_i}\}$ be a minimal set of generators of the multi-graded part of $(\mathfrak{a}_1, \ldots, \mathfrak{a}_m, \mathcal{J})$ of multi-degree*

$$(0, \ldots, \underbrace{1}_{i\text{-}th}, \ldots, 0, *),$$

where $*$ denotes arbitrary degree in $\mathbf{y}$. We denote by ψ_i the Jacobian matrix of the collection of polynomials $\{h_{i,1}, \ldots, h_{i,k_i}\}$ with respect to $\mathbf{x}_i$, that is

$$\psi_i = \begin{pmatrix} \frac{h_{i,1}}{\partial x_{i,0}} & \frac{h_{i,1}}{\partial x_{i,1}} & \cdots & \frac{h_{i,1}}{\partial x_{i,r_i}} \\ \frac{h_{i,2}}{\partial x_{i,0}} & \frac{h_{i,2}}{\partial x_{i,1}} & \cdots & \frac{h_{i,2}}{\partial x_{i,r_i}} \\ \vdots & \vdots & & \vdots \\ \frac{h_{i,k_i}}{\partial x_{i,0}} & \frac{h_{i,k_i}}{\partial x_{i,1}} & \cdots & \frac{h_{i,k_i}}{\partial x_{i,r_i}} \end{pmatrix}.$$

Following [16, 46, 129]*, the matrix ψ_i will be called the $\mathbf{x}_i$-partial Jacobian dual matrix. We note that its entries are polynomials in $\mathbb{k}[\mathbf{y}]$. Finally, the matrix obtained by concatenating all the ψ_i's in the main diagonal*

$$\psi = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \psi_m \end{pmatrix}$$

will be called the full Jacobian dual matrix.

The next proposition is based on [46, Proposition 2.15]. It shows that the ranks of the Jacobian dual matrices are sensitive to the dimensions of the source and the target.

Proposition 3.38. *Let $\mathcal{F} : X_1 \times \cdots \times X_m \dashrightarrow Y$ be a dominant rational map. Then, we have the following inequalities:*

$$\dim(X_1) + \cdots + \dim(X_m) - \dim(Y) \leqslant \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \operatorname{rank}_S(\psi_i \otimes_{\mathbb{k}[\mathbf{y}]} S). \tag{3.12}$$

$$\operatorname{rank}_S(\psi_i \otimes_{\mathbb{k}[\mathbf{y}]} S) \leqslant r_i \text{ for each } i = 1, \ldots, m. \tag{3.13}$$

Proof. We begin with the first inequality. For each $1 \leqslant i \leqslant m$. Let E_i be the S-module $E_i = \operatorname{Coker}_S(\psi_i^t \otimes_{\mathbb{k}[\mathbf{y}]} S)$ with presentation

$$S^{k_i} \xrightarrow{\psi_i^t \otimes_{\mathbb{k}[\mathbf{y}]} S} S^{r_i+1} \to E_i \to 0.$$

The direct sum $E = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ is an S-module with presentation

$$S^{k_1} \oplus S^{k_2} \oplus \cdots \oplus S^{k_m} \xrightarrow{\psi^t \otimes_{\Bbbk[\mathbf{y}]} S} S^{r_1+1} \oplus S^{r_2+1} \oplus \cdots \oplus S^{r_m+1} \to E \to 0.$$

By the definition of the Jacobian dual matrices we have that $I_1(\mathbf{x} \cdot \psi^t) \subset \mathcal{J}$, and we saw in the proof of Lemma 3.36 that $\mathfrak{b} \subset \mathcal{J}$. Hence, we get a canonical surjective homomorphism $\mathrm{Sym}_S(E) \twoheadrightarrow \mathcal{R}_R(I)$ of S-algebras given by

$$\begin{aligned}\alpha : \mathrm{Sym}_S(E) &\cong S[\mathbf{x}]/I_1\big(\mathbf{x} \cdot \big(\psi^t \otimes_{\Bbbk[\mathbf{y}]} S\big)\big) \cong \Bbbk[\mathbf{y}][\mathbf{x}]/\big(\mathfrak{b}, I_1(\mathbf{x} \cdot \psi^t)\big) \twoheadrightarrow R[\mathbf{y}]/\big(\mathfrak{b}, I_1(\mathbf{x} \cdot \psi^t)\big)\\ &\twoheadrightarrow \mathcal{R}_R(I).\end{aligned}$$

Following [132], we have that $\mathcal{R}_S(E) = \mathrm{Sym}_S(E)/\mathcal{T}$ where $\mathcal{T}$ represents the S-torsion submodule of $\mathrm{Sym}_S(E)$. Let $G \in \mathcal{T}$, there exists some $s \in S \setminus 0$ such that $s \cdot G = 0 \in \mathrm{Sym}_S(E)$. By using the isomorphisms $S \cong \Bbbk[\mathbf{f}] \cong \Bbbk[\mathbf{f}t] \subset \mathcal{R}_R(I)$, we can see that

$$0 = \alpha(s \cdot G) = \alpha(s)\alpha(G) = s(\mathbf{f}t)\alpha(G) \in \mathcal{R}_R(I)$$

where $s(\mathbf{f}t) \neq 0$. Since $\mathcal{R}_R(I)$ is an integral domain then it follows that $\alpha(G) = 0$, and so we have a canonical surjective homomorphism

$$\mathcal{R}_S(E) \twoheadrightarrow \mathcal{R}_R(I) \tag{3.14}$$

of S-algebras.

Finally, from Theorem 1.27 we get

$$\begin{aligned}\dim\big(\mathcal{R}_R(I)\big) &\leqslant \dim\big(\mathcal{R}_S(E)\big)\\ \dim(R) + 1 &\leqslant \dim(S) + m + \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \mathrm{rank}_S(\psi_i \otimes_{\Bbbk[\mathbf{y}]} S),\end{aligned}$$

and using the equality $\dim(R) = \dim(A_1) + \cdots + \dim(A_m)$, we substitute

$$\begin{aligned}\dim(X_1) + \cdots + \dim(X_m) + m + 1 &\leqslant \dim(Y) + 1 + m + \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \mathrm{rank}_S(\psi_i \otimes_{\Bbbk[\mathbf{y}]} S)\\ \dim(X_1) + \cdots + \dim(X_m) - \dim(Y) &\leqslant \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \mathrm{rank}_S(\psi_i \otimes_{\Bbbk[\mathbf{y}]} S).\end{aligned}$$

Now, we turn to the proof of the second claimed inequality. We follow one of the steps in the proof of [16, Proposition 3]. Fix $i = 1, \ldots, m$. We have that $A_1 \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_{i-1} \otimes_{\Bbbk} A_{i+1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_m$ is an integral domain, and let us denote by $\mathbb{L}$ its quotient field. Let $X_i' = \mathrm{Proj}\big(\mathbb{L}[\mathbf{x}_i]/(\mathfrak{a}_i)\big)$, we define

a rational map
$$\mathcal{F}' : X_i' \dashrightarrow Y' = \mathrm{Proj}(S') \subset \mathbb{P}^s_{\mathbb{L}}$$
given by the classes of $f_0, \ldots, f_s$ inside $\mathbb{L}[\mathbf{x}_i]/(\mathfrak{a}_i)$, and we denote $S' := \mathbb{L}[\mathbf{f}]$. Using the field inclusion $\mathbb{k} \hookrightarrow \mathbb{L}$ we can check that any polynomial in the defining equations of the Rees algebra $\mathcal{R}_R(I)$ is also contained in the defining equations of the Rees algebra $\mathcal{R}_{\mathbb{L}[\mathbf{x}_i]/(\mathfrak{a}_i)}(I)$. In particular, we have that the row space of $\psi_i \otimes_{\mathbb{k}[\mathbf{y}]} S$ is contained in the row space of $\psi' \otimes_{\mathbb{L}[\mathbf{y}]} S'$, where ψ' denotes the Jacobian dual matrix of $\mathcal{F}'$. Hence, $\mathrm{rank}_S(\psi_i \otimes_{\mathbb{k}[\mathbf{y}]} S) \leqslant \mathrm{rank}_{S'}(\psi' \otimes_{\mathbb{L}[\mathbf{y}]} S') \leqslant r_i$, and the last inequality follows from [46, Corollary 2.16] or (3.12). $\square$

The following birationality criterion is the main result of this section; it is the multi-graded version of [46, Theorem 2.18] and [16, Theorem 2].

Theorem 3.39. *Let* $\mathcal{F} : X_1 \times \cdots \times X_m \dashrightarrow Y$ *be a dominant rational map. Then, the following three conditions are equivalent:*

(i) $\mathcal{F}$ *is birational.*

(ii) $\mathrm{rank}_S(\psi_i \otimes_{\mathbb{k}[\mathbf{y}]} S) = r_i$ *for each* $i = 1, \ldots, m$.

(iii) $\mathrm{rank}_S(\psi \otimes_{\mathbb{k}[\mathbf{y}]} S) = r_1 + r_2 + \cdots + r_m$.

In addition, if $\mathcal{F}$ *is birational then its inverse is of the form* $\mathcal{G} : Y \dashrightarrow X_1 \times \cdots \times X_m$*, where each map* $Y \dashrightarrow X_i$ *is given by the signed ordered maximal minors of an* $r_i \times (r_i + 1)$ *submatrix of* ψ_i *of rank* r_i.

Proof. $(i) \Rightarrow (ii)$. Let us suppose that $\mathcal{F}$ is birational. From Lemma 3.36 we get an isomorphism $\mathcal{R}_R(I) \cong \mathcal{R}_S(J_1 \oplus \cdots \oplus J_m)$ induced by the identity of $\mathbb{k}[\mathbf{x}, \mathbf{y}]$. So we obtain the equality $(\mathcal{I}, \mathfrak{a}_1, \ldots, \mathfrak{a}_m) = (\mathcal{J}, \mathfrak{b})$ that in particular gives us
$$[(\mathcal{I}, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)]_{(\underbrace{0,\ldots,1,\ldots,0}_{\mathbf{x}}, \underbrace{*}_{\mathbf{y}})} = [(\mathcal{J}, \mathfrak{b})]_{(\underbrace{*}_{\mathbf{y}}, \underbrace{0,\ldots,1,\ldots,0}_{\mathbf{x}})} \tag{3.15}$$
for each $i = 1, \ldots, m$. By reducing modulo $\mathfrak{b}$, the right hand side of (3.15) yields a presentation
$$0 \to [(\mathcal{J}, \mathfrak{b})/\mathfrak{b}]_{(*,0,\ldots,1,\ldots,0)} \to S[\mathbf{x}_i] \to \mathrm{Sym}_S(\mathbf{g}_i) \to 0$$
of the symmetric algebra $\mathrm{Sym}_S(\mathbf{g}_i)$ of $\mathbf{g}_i$. On the other hand, from the definition of Jacobian dual matrices we have
$$[(\mathcal{I}, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)/\mathfrak{b}]_{(0,\ldots,1,\ldots,0,*)} = I_1\big(\mathbf{x}_i \cdot (\psi_i^t \otimes_{\mathbb{k}[\mathbf{y}]} S)\big).$$
Let $\mathrm{Syz}_S(\mathbf{g}_i)$ be the matrix of syzygies of $\mathbf{g}_i$. By the two previous reductions of (3.15), we obtain
$$I_1\big(\mathbf{x}_i \cdot (\psi_i^t \otimes_{\mathbb{k}[\mathbf{y}]} S)\big) = I_1(\mathbf{x}_i \cdot \mathrm{Syz}_S(\mathbf{g}_i)). \tag{3.16}$$

Since both matrices $\psi_i^t \otimes_{\Bbbk[\mathbf{y}]} S$ and $\mathrm{Syz}_S(\mathbf{g}_i)$ have entries in S, the column space of $\mathrm{Syz}_S(\mathbf{g}_i)$ is equal to the one of $\psi_i^t \otimes_{\Bbbk[\mathbf{y}]} S$. Finally, the fact that $\mathrm{rank}_S(\mathrm{Syz}_S(\mathbf{g}_i)) = r_i$ implies that $\mathrm{rank}_S(\psi_i^t \otimes_{\Bbbk[\mathbf{y}]} S) = r_i$.

(ii) $\Rightarrow$ (i). We assume that $\mathrm{rank}_S(\psi_i \otimes_{\Bbbk[\mathbf{y}]} S) = r_i$ for each $i = 1, \ldots, m$. Let $i = 1, \ldots, m$. Let M_i be a $r_i \times (r_i + 1)$ submatrix of ψ_i such that $\mathrm{rank}_S\left(M_i \otimes_{\Bbbk[\mathbf{y}]} S\right) = r_i$. Denote by $\Delta_0(\mathbf{y}), \Delta_1(\mathbf{y}), \cdots, \Delta_{r_i}(\mathbf{y})$ the ordered signed minors of M_i^t. The Hilbert-Koszul lemma ([46, Propositon 2.1]) implies that the vector $e_a\Delta_b(\mathbf{y}) - e_b\Delta_a(\mathbf{y})$ belongs to the column space of M_i^t, and so also to the one of ψ_i^t. Since by definition $I_1(\mathbf{x}_i \cdot \psi_i^t) = [(\mathcal{J}, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)]_{(0,\ldots,1,\ldots,0,*)}$, we get $x_{i,a}\Delta_b(\mathbf{y}) - x_{i,b}\Delta_a(\mathbf{y}) \in [(\mathcal{J}, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)]_{(0,\ldots,1,\ldots,0,*)}$.

Making a substitution via the canonical homomorphism $\Bbbk[\mathbf{x}, \mathbf{y}] \to \mathcal{R}_R(I)$, we automatically get

$$x_{i,a}\Delta_b(\mathbf{f}) - x_{i,b}\Delta_a(\mathbf{f}) = 0 \in R, \quad \text{for every pair } a, b.$$

From the inclusion $S \cong \Bbbk[\mathbf{f}] \cong \Bbbk[\mathbf{f}t] \subset \mathcal{R}_R(I)$ and the rank assumption, we have that the tuple

$$(\Delta_0(\mathbf{f}), \ldots, \Delta_{r_i}(\mathbf{f}))$$

does not vanish on R. Let $\mathcal{G} : Y \dashrightarrow X_1 \times \cdots \times X_m$ where each map $Y \to X_i$ is given by the tuple $(\Delta_0(\mathbf{y}), \ldots, \Delta_{r_i}(\mathbf{y})) \otimes_{\Bbbk[\mathbf{y}]} S$. We have proven that $\mathcal{G}$ is the inverse of $\mathcal{F}$.

(ii) $\Leftrightarrow$ (iii). This part follows from the inequalities of (3.13) and the fact that $\mathrm{rank}_S(\psi \otimes_{\Bbbk[\mathbf{y}]} S) = \sum_{i=1}^m \mathrm{rank}_S(\psi_i \otimes_{\Bbbk[\mathbf{y}]} S)$. $\square$

To illustrate this theorem, we provide two corollaries. The first one is a rigorous translation of birationality in terms of an isomorphism between the corresponding Rees algebras; this result is the multi-graded version of [129, Proposition 2.1]. The second is a specific birationality criterion dedicated to some particular monomial maps.

Corollary 3.40. *The rational map $\mathcal{F} : X = X_1 \times \cdots \times X_m \dashrightarrow Y$ is birational with inverse $\mathcal{G}$ if and only if $\mathcal{F}$ is dominant, the image of $\mathcal{G}$ is X, and the identity map of $\Bbbk[\mathbf{x}, \mathbf{y}]$ induces a $\Bbbk$-algebra isomorphism between the Rees algebra $\mathcal{R}_R(I)$ and the multi-graded Rees algebra $\mathcal{R}_S(J_1 \oplus J_2 \oplus \cdots \oplus J_m)$.*

Proof. One implication was proved in Lemma 3.36. Let us assume that $\mathcal{F}$ and $\mathcal{G}$ are dominant and the identity map of $\Bbbk[\mathbf{x}, \mathbf{y}]$ induces an isomorphism between $\mathcal{R}_R(I)$ and $\mathcal{R}_S(J_1 \oplus \cdots \oplus J_m)$.

As in Proposition 3.38, let $E = \mathrm{Coker}_S(\psi^t \otimes_{\Bbbk[\mathbf{y}]} S)$. Identity (3.16) gives us a canonical isomorphism of S-algebras

$$\mathrm{Sym}_S(E) \cong \mathrm{Sym}_S(J_1 \oplus \cdots \oplus J_m).$$

From the assumption $\mathcal{R}_S(J_1 \oplus \cdots \oplus J_m) \cong \mathcal{R}_R(I)$, we get the following isomorphisms

$$\mathcal{R}_S(E) \cong \mathcal{R}_S(J_1 \oplus \cdots \oplus J_m) \cong \mathcal{R}_R(I),$$

which are induced by the identity map on $\Bbbk[\mathbf{x}, \mathbf{y}]$.

Performing the same computation of Proposition 3.38, now we get

$$\dim(S) + \sum_{i=1}^{m}(r_i + 1) - \operatorname{rank}_S(\psi \otimes_{\Bbbk[\mathbf{y}]} S) = \dim(R) + 1$$

$$\dim(Y) + m + 1 + \sum_{i=1}^{m} r_i - \operatorname{rank}_S(\psi \otimes_{\Bbbk[\mathbf{y}]} S) = \dim(X_1) + \cdots + \dim(X_m) + m + 1.$$

Since $\mathcal{F}$ and $\mathcal{G}$ are dominant, we have $\dim(Y) = \dim(X_1) + \cdots + \dim(X_m)$. So it follows that $\operatorname{rank}_S(\psi \otimes_{\Bbbk[\mathbf{y}]} S) = \sum_{i=1}^{m} r_i$.

Therefore, from Theorem 3.39 we have that $\mathcal{F}$ is birational. Let us denote by $\mathcal{G}'$ its inverse. Let $J'_1, \ldots, J'_m$ be the base ideals of $\mathcal{G}'$. Applying Lemma 3.36, we have that the identity map of $\Bbbk[\mathbf{x}, \mathbf{y}]$ induces the following isomorphisms

$$\mathcal{R}_S(J'_1 \oplus \cdots \oplus J'_m) \cong \mathcal{R}_R(I) \cong \mathcal{R}_S(J_1 \oplus \cdots \oplus J_m).$$

In particular, we have an isomorphism between the symmetric algebras of $J_1 \oplus \cdots \oplus J_m$ and $J'_1 \oplus \cdots \oplus J'_m$ over S, which implies an equality between their syzygies. Therefore, the tuples defining $\mathcal{G}$ and $\mathcal{G}'$ are proportional and so they define the same rational map. $\square$

Now, we focus on the case of a monomial multi-graded rational map $\mathcal{F} : \left(\mathbb{P}^1\right)^s \dashrightarrow \mathbb{P}^s$. We assume that $I = (\mathbf{x}^{\alpha_0}, \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_s})$, where each $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,2s})$ is a vector of $2s$ entries, and $\mathbf{x}^{\alpha_i}$ denotes the monomial

$$\mathbf{x}^{\alpha_i} = x_{1,0}^{\alpha_{i,1}} x_{1,1}^{\alpha_{i,2}} \cdots x_{s,0}^{\alpha_{i,2s-1}} x_{s,1}^{\alpha_{i,2s}}.$$

In this setting, the presentation (3.4) of $\mathcal{R}(I)$ can be encoded by the following matrix:

$$M = \begin{pmatrix} e_1 & e_2 & \ldots & e_{2s} & \alpha_{0,1} & \alpha_{1,1} & \ldots & \alpha_{s,1} \\ & & & & \vdots & \vdots & \vdots & \vdots \\ & & & & \alpha_{0,2s} & \alpha_{1,2s} & \ldots & \alpha_{s,2s} \\ & & & & 1 & 1 & \ldots & 1 \end{pmatrix}, \tag{3.17}$$

where $e_1, e_2, \ldots e_{2s}$ are the first $2s$ unit vectors in $\mathbb{Z}^{2s+1}$. For any integer vector $\beta \in \mathbb{Z}^{3s+1}$, we denote by $\mathbf{xy}^{\beta}$ the following monomial

$$\mathbf{xy}^{\beta} = x_{1,0}^{\beta_1} x_{1,1}^{\beta_2} \cdots x_{s,0}^{\beta_{2s-1}} x_{s,1}^{\beta_{2s}}\, y_0^{\beta_{2s+1}} y_1^{\beta_{2s+2}} \cdots y_s^{\beta_{3s+1}}.$$

The ideal of defining equations of $\mathcal{R}(I)$ is a toric ideal (see [138, Chapter 4]). It is generated by the following binomials

$$\mathcal{I} = \left(\mathbf{x}\mathbf{y}^{\beta^+} - \mathbf{x}\mathbf{y}^{\beta^-} \mid M\beta = 0, \;\; \beta = \beta^+ - \beta^-, \;\; \beta^+, \beta^- \geqslant 0\right). \tag{3.18}$$

The following corollary contains a very effective way of testing the birationality of $\mathcal{F}$, which can be done for instance by using Hermite normal form algorithms.

Corollary 3.41. *Let $\mathcal{F} : \left(\mathbb{P}^1\right)^s \dashrightarrow \mathbb{P}^s$ be a monomial dominant multi-graded rational map. Let A be the submatrix of M in (6.3) given by the last $s+1$ columns. Then, $\mathcal{F}$ is birational if and only if the following conditions are satisfied for each $i = 1, \ldots, s$:*

$$\left\{\gamma \in \mathbb{Z}^{s+1} \;\mid\; A\gamma = e_{2i-1} - e_{2i}\right\} \neq \emptyset. \tag{3.19}$$

Proof. From Theorem 3.39 we only need to check that

$$\mathcal{I}_{(0,\ldots,\underbrace{1}_{i\text{-th}},\ldots,0,*)} \neq 0 \quad \text{for each} \quad i = 1, \ldots, m.$$

By the description of (3.18), this inequality is equivalent to the solution of the systems of equations given in (3.19). ☐

Linear syzygies and some consequences

The birationality criterion provided in Theorem 3.39 requires the computation of the equations of the Rees algebra of the base ideal of a rational map. In this subsection, we investigate how the syzygies of the base ideal can be used instead in order to deduce, or to characterize, the birationality of a multi-graded rational map.

Notation 3.42. *Let φ be the matrix of syzygies of I whose entries are multi-homogeneous polynomials. We denote by φ_1 the submatrix of φ whose columns are the columns of ψ corresponding to syzygies of I of multi-degree $(1, \ldots, 0)$, $(0, 1, 0, \ldots)$,..., or $(0, \ldots, 1)$. The matrix φ_1 is called the linear part of the matrix φ.*

The following proposition is based on [46, Theorem 3.2] and [16, Proposition 3].

Proposition 3.43. *Let $\mathcal{F} : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_m} \dashrightarrow \mathbb{P}^{r_1+\cdots+r_m}$ be a dominant rational map. If* $\operatorname{rank}(\varphi_1) = r_1 + \cdots + r_m$, *then $\mathcal{F}$ is birational.*

Proof. We choose a matrix ρ with entries in S such that $\mathbf{y} \cdot \varphi_1 = \mathbf{x} \cdot \rho$. Let $E = \operatorname{Coker}(\rho)$, then the previous equality gives us the isomorphism $\operatorname{Sym}_R(\operatorname{Coker}(\varphi_1)) \cong \operatorname{Sym}_S(E)$. We present the Rees algebras $\mathcal{R}_R(\operatorname{Coker}(\varphi_1))$ and $\mathcal{R}_S(E)$ by

$$\mathcal{R}_R(\operatorname{Coker}(\varphi_1)) = \frac{\mathbb{k}[\mathbf{x}, \mathbf{y}]}{(I_1(\mathbf{y} \cdot \varphi_1), \mathcal{T}_1)} \quad \text{and} \quad \mathcal{R}_S(E) = \frac{\mathbb{k}[\mathbf{x}, \mathbf{y}]}{(I_1(\mathbf{x} \cdot \rho), \mathcal{T}_2)},$$

where $\mathcal{T}_1$ represents the R-torsion of $\mathrm{Sym}_R(\mathrm{Coker}(\varphi_1))$ and $\mathcal{T}_2$ is the S-torsion of $\mathrm{Sym}_S(E)$, both lifted to $\Bbbk[\mathbf{x},\mathbf{y}]$. Since S is an integral domain and E has rank, then $\mathcal{R}_S(E)$ is an integral domain and so $(I_1(\mathbf{x}\cdot\rho),\mathcal{T}_2)$ is a prime ideal.

Let $G(\mathbf{x},\mathbf{y}) \in \mathcal{T}_1$. There exists $F(\mathbf{x}) \in \Bbbk[\mathbf{x}] \setminus 0$ such that $F(\mathbf{x})G(\mathbf{x},\mathbf{y}) \in I_1(\mathbf{y}\cdot\varphi_1) \subset (I_1(\mathbf{x}\cdot\rho),\mathcal{T}_2)$. We assume $G(\mathbf{x},\mathbf{y}) \notin (I_1(\mathbf{x}\cdot\rho),\mathcal{T}_2)$, then it follows that $F(\mathbf{x}) \in \mathcal{T}_2$ due to the fact that $(I_1(\mathbf{x}\cdot\rho),\mathcal{T}_2)$ is prime and the ideal $I_1(\mathbf{x}\cdot\rho)$ is generated by multi-homogeneous polynomials with positive degree on $\mathbf{y}$. Thus, there exists a polynomial $H(\mathbf{y}) \in \Bbbk[\mathbf{y}]\setminus 0$ such that $H(\mathbf{y})F(\mathbf{x}) \in I_1(\mathbf{x}\cdot\rho)$. Since $I_1(\mathbf{x}\cdot\rho)$ is generated by syzygies of I, when we substitute $y_j \mapsto f_j$, we get $H(\mathbf{f})F(\mathbf{x}) = 0$. From the fact that $H(\mathbf{f}) \neq 0$ (note that here we have $S \cong \Bbbk[\mathbf{y}]$), it follows the contradiction $F(\mathbf{x}) = 0$.

Therefore, we have a surjective R-algebra map $\mathcal{R}_R(\mathrm{Coker}(\varphi_1)) \twoheadrightarrow \mathcal{R}_S(E)$, and so we get the inequality

$$\dim(\mathcal{R}_S(E)) \leqslant \dim(\mathcal{R}_R(\mathrm{Coker}(\varphi_1)))$$
$$\dim(S) + \sum_{i=1}^{m}(r_i+1) - \mathrm{rank}(\rho) \leqslant \dim(R) + 1 + \sum_{i=1}^{m} r_i - \mathrm{rank}(\varphi_1).$$

Substituting $\mathrm{rank}(\varphi_1) = \sum_{i=1}^m r_i$, $\dim(S) = 1 + \sum_{i=1}^m r_i$ and $\dim(R) = \sum_{i=1}^m (r_i+1)$, we get

$$\sum_{i=1}^{m} r_i \leqslant \mathrm{rank}(\rho).$$

The inclusion $I_1(\mathbf{x}\cdot\rho) \subset I_1\left(\mathbf{x}\cdot\psi^t\right)$ gives us the inequality $\mathrm{rank}(\rho) \leqslant \mathrm{rank}(\psi^t)$. Combining this with Proposition 3.38 we necessarily get $\mathrm{rank}_S(\psi \otimes_{\Bbbk[\mathbf{y}]} S) = \sum_{i=1}^m r_i$. Therefore, the result follows from Theorem 3.39. □

The above proposition gives a sufficient syzyzy-based property to ensure birationality. In the next result we prove that it becomes also a necessary condition under the assumption that the base ideal is of linear type. This effective birationality criterion is the multi-graded version of [46, Proposition 3.4].

Theorem 3.44. *Let $\mathcal{F} : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_m} \dashrightarrow \mathbb{P}^{r_1+\cdots+r_m}$ be a rational map whose base ideal $I = (\mathbf{f})$ is of linear type. Then, the following conditions are equivalent:*

(i) *$\mathcal{F}$ is birational.*

(ii) $\mathrm{rank}(\varphi_1) = r_1 + \cdots + r_m$.

To prove this theorem, we will need the following preliminary lemma on the torsion of symmetric algebras in the multi-graded setting. It is essentially an adaptation of Lemma 1.10 to the multi-graded case. As we are following the general setup of [132], $\mathcal{R}_R(I_1 \oplus \cdots \oplus I_n)$ means $\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)$ modulo its R-torsion.

Lemma 3.45. *Let* R *be a Noetherian commutative ring and* $I_1, \ldots, I_n$ *be ideals having rank. Then, we have the following relation between (multi-graded) symmetric and Rees algebras*

$$\mathcal{R}_R(I_1 \oplus \cdots \oplus I_n) = \frac{\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)}{H^0_{I_1 \cdots I_n}\big(\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)\big)}.$$

In particular, if R *is local with maximal ideal* $\mathfrak{m}$ *and each* I_i *is* $\mathfrak{m}$*-primary then we have*

$$\mathcal{R}_R(I_1 \oplus \cdots \oplus I_n) = \frac{\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)}{H^0_{\mathfrak{m}}\big(\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)\big)}.$$

Proof. As part of the proof of this lemma we shall obtain that $\mathcal{R}_R(I_1 \oplus \cdots \oplus I_n)$ coincides with the usual multi-graded Rees algebra

$$\mathcal{R}_R(I_1, \ldots, I_n) = \bigoplus_{i_1,\ldots,i_n} I_1^{i_1} \cdots I_n^{i_n} t_1^{i_1} \cdots t_n^{i_n}.$$

By the assumption that each ideal I_i has rank then we have $\mathrm{grade}(I_i) \geqslant 1$ (see e.g. [19, proof of Corollary 1.4.7]), and from the Unmixedness Theorem (see e.g. [19, Exercise 1.2.21], [95, Theorem 125]) we can assume that $I_i = (\mathbf{f}_i)$ where $\mathbf{f}_i = (f_{i,1}, \ldots, f_{i,m_i})$ and each $f_{i,j}$ is an R-regular element.

Let $\mathfrak{A}$ be the polynomial ring $\mathfrak{A} = R[\mathbf{T}_1, \ldots, \mathbf{T}_n]$ where $\mathbf{T}_i = \{T_{i,1}, \ldots, T_{i,m_i}\}$ for each $i = 1, \ldots, n$. The symmetric algebra $\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)$ can easily be presented by

$$0 \rightarrow \mathcal{J}_1 \rightarrow \mathfrak{A} \rightarrow \mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n) \rightarrow 0,$$

where $\mathcal{J}_1 = \big(I_1(\mathbf{T}_1 \cdot \mathrm{Syz}(\mathbf{f}_i)), \ldots, I_1(\mathbf{T}_n \cdot \mathrm{Syz}(\mathbf{f}_n))\big)$. On the other hand, the Rees algebra can be presented by

$$\begin{array}{c} 0 \rightarrow \mathcal{J} \rightarrow \mathfrak{A} \rightarrow \mathcal{R}_R(I_1, \ldots, I_n) \rightarrow 0 \\ \mathbf{T}_i \mapsto \mathbf{f}_i t_i, \end{array}$$

where $\mathcal{J}$ is the ideal generated by the multi-homogeneous polynomials $F(\mathbf{T}_1, \ldots, \mathbf{T}_n) \in \mathfrak{A}$ such that $F(\mathbf{f}_1, \ldots, \mathbf{f}_n) = 0$. Therefore, we want to analyze the canonical exact sequence

$$0 \rightarrow (\mathcal{J}/\mathcal{J}_1) \rightarrow \mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n) \xrightarrow{\alpha} \mathcal{R}_R(I_1, \ldots, I_n) \rightarrow 0.$$

It is clear that the R-torsion submodule of $\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)$ is contained in $\mathrm{Ker}(\alpha)$, and in particular, by the assumption on the ideals I_i, the elements of $\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)$ annihilated by some power $(I_1 \ldots I_n)^l$ are also contained in $\mathrm{Ker}(\alpha)$. If we prove that any element in $\mathrm{Ker}(\alpha)$ is contained in the R-torsion submodule of $\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)$ and is annihilated by some power

$(I_1 \cdots I_n)^l$, then we are done because we get the following equality and isomorphism

$$\mathcal{R}_R(I_1 \oplus \cdots \oplus I_n) = \frac{\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)}{H^0_{I_1 \cdots I_n}\big(\mathrm{Sym}_R(I_1 \oplus \cdots \oplus I_n)\big)} \cong \mathcal{R}_R(I_1, \ldots, I_n).$$

By the assumption that all the $f_{i,j}$ are R-regular, the proof of the two previous assertions will follow from the next claim.

Claim. Let $F \in \mathcal{J}$. Then, for any element of the form $f_{1,j_1} f_{2,j_2} \cdots f_{n,j_n}$ (i.e. a generator of $I_1 \cdots I_n$), there exists some integer $l > 0$ such that $(f_{1,j_1} f_{2,j_2} \cdots f_{n,j_n})^l F \in \mathcal{J}_1$.

Proof of the claim. Fix any generators $f_{1,j_1} \in I_1, f_{2,j_2} \in I_2, \ldots, f_{n,j_n} \in I_n$. Let $F \in \mathcal{J}$ be multi-homogeneous of multi-degree $(d_1, d_2, \ldots, d_n)$ we shall proceed by induction on $d = d_1 + \cdots + d_n$. In the inductive step, it is enough to prove that there exists integers $\alpha_1 \geqslant 0, \ldots, \alpha_n \geqslant 0$ such that

$$f^{\alpha_1}_{1,j_1} f^{\alpha_2}_{2,j_2} \cdots f^{\alpha_n}_{n,j_n} F \in \mathcal{J}_1.$$

If $d = 1$ then F clearly satisfies the previous condition. So, we assume that $d > 1$ and by simply ordering the variables $\mathbf{T}_i$ we may suppose that $d_1 \geqslant 1$. We can write F in the following way

$$F = \sum_{k=1}^{m_1} T_{1,k} H_k\Big(T_{1,k}, \ldots, T_{1,m_1}, \mathbf{T}_2, \ldots, \mathbf{T}_n\Big)$$

Then we define the following polynomial

$$G = \sum_{k=1}^{m_1} T_{1,k} H_k\Big(f_{1,k}, \ldots, f_{1,m_1}, \mathbf{f}_2, \ldots, \mathbf{f}_n\Big)$$

which belong $\mathcal{J}_1$. We compute

$$\begin{aligned} f^{d_1-1}_{1,j_1} f^{d_2}_{2,j_2} \cdots f^{d_n}_{n,j_n} F - T^{d_1-1}_{1,j_1} T^{d_2}_{2,j_2} \cdots T^{d_n}_{n,j_n} G = \\ \sum_{k=1}^{m_1} T_{1,k} \Bigg(f^{d_1-1}_{1,j_1} f^{d_2}_{2,j_2} \cdots f^{d_n}_{n,j_n} H_k\Big(T_{1,k}, \ldots, T_{1,m_1}, \mathbf{T}_2, \ldots, \mathbf{T}_n\Big) - \\ T^{d_1-1}_{1,j_1} T^{d_2}_{2,j_2} \cdots T^{d_n}_{n,j_n} H_k\Big(f_{1,k}, \ldots, f_{1,m_1}, \mathbf{f}_2, \ldots, \mathbf{f}_n\Big)\Bigg), \end{aligned}$$

where each polynomial

$$\begin{aligned} f^{d_1-1}_{1,j_1} f^{d_2}_{2,j_2} \cdots f^{d_n}_{n,j_n} H_k\Big(T_{1,k}, \ldots, T_{1,m_1}, \mathbf{T}_2, \ldots, \mathbf{T}_n\Big) \\ - T^{d_1-1}_{1,j_1} T^{d_2}_{2,j_2} \cdots T^{d_n}_{n,j_n} H_k\Big(f_{1,k}, \ldots, f_{1,m_1}, \mathbf{f}_2, \ldots, \mathbf{f}_n\Big) \end{aligned}$$

belongs to $\mathcal{J}$ and has total degree smaller than d. Therefore, the proof of the claim follows by induction. □

Proof of Theorem 3.44. $(ii) \Rightarrow (i)$ Since I is of linear type, the polynomials of $\mathbf{f}$ are algebraically independent. Therefore, this implication follows from Proposition 3.43.

$(i) \Rightarrow (ii)$ From the assumption of $\mathcal{F}$ being birational, let $\mathbf{g}_1, \ldots, \mathbf{g}_m$ be a set of representatives of the inverse map $\mathcal{G} : \mathbb{P}^{r_1+\cdots+r_m} \dashrightarrow \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_m}$.

Since I is of linear type, we have $\mathcal{J} = I_1(\mathbf{y} \cdot \varphi)$ and so we obtain the following equality

$$I_1(\mathbf{y} \cdot \varphi_1) = I_1\left(\mathbf{x} \cdot \psi^t\right). \tag{3.20}$$

Due to the isomorphism obtained in Lemma 3.36, the module $(\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)$ has the following presentation

$$S^p \xrightarrow{\psi^t} S^{r_1+\cdots+r_m+m} \to (\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m) \to 0.$$

We also also consider the module $E = \mathrm{Coker}(\varphi_1)$ with presentation

$$R^p \xrightarrow{\varphi_1} R^{r_1+\cdots r_m+1} \to E \to 0.$$

From the equality (3.20) we get an isomorphism $\mathrm{Sym}_S\big((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)\big) \cong \mathrm{Sym}_R(E)$ induced by the identity map of $\mathbb{k}[\mathbf{x}, \mathbf{y}]$. Then, we have the following

$$\mathrm{Sym}_S\big((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)\big) \cong \mathrm{Sym}_R(E) \twoheadrightarrow \mathcal{R}_R(I).$$

Let $\mathcal{T}$ be the S-torsion of $\mathrm{Sym}_S\big((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)\big)$ and λ be the isomorphism

$$\lambda : \mathrm{Sym}_S\big((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)\big) \xrightarrow{\cong} \mathrm{Sym}_R(E)$$

If we prove that $\lambda(\mathcal{T})$ is contained in the R-torsion of $\mathrm{Sym}_R(E)$, we will get the following epimorphims

$$\mathcal{R}_S\big((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)\big) \twoheadrightarrow \mathcal{R}_R(E) \twoheadrightarrow \mathcal{R}_R(I).$$

Therefore, from Lemma 3.36 we get $\mathcal{R}_S((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)) \cong \mathcal{R}_R(E) \cong \mathcal{R}_R(I)$ and so $\mathrm{rank}(E) = 1$ which implies the statement.

Thus we shall focus on the claim below:

Claim: $\lambda(\mathcal{T})$ is contained in the R-torsion of $\mathrm{Sym}_R(E)$.

Proof of the claim. First, by applying Lemma 3.45 we get that there exists some l such that $\big((\mathbf{g}_1) \cdots (\mathbf{g}_m)\big)^l \mathcal{T} = 0$. Here, we are considering $(\mathbf{g}_1) \cdots (\mathbf{g}_m) \subset S \subset \mathrm{Sym}_S\big((\mathbf{g}_1) \oplus \cdots \oplus (\mathbf{g}_m)\big)$, thus $\big((\mathbf{g}_1) \cdots (\mathbf{g}_m)\big)^l$ lifts to $\mathbb{k}[\mathbf{x}, \mathbf{y}]$ exactly as $\big((\mathbf{g}_1) \cdots (\mathbf{g}_m)\big)^l$. We map into $\mathrm{Sym}_R(E)$ using the canonical map

$$\begin{aligned} \mathbb{k}[\mathbf{x}, \mathbf{y}] &\to \mathrm{Sym}_R(E) \\ \mathbf{x}_i \mapsto \mathbf{x}_i, &\qquad y_i \mapsto e_i, \end{aligned}$$

where e_i are the homogeneous generators of E given by its presentation. Summarizing, we have that

$$\lambda\Big(\big((\mathbf{g}_1)\cdots(\mathbf{g}_m)\big)^l\mathcal{T}\Big) = \big((\mathbf{g}_1(\mathbf{e}))\cdots(\mathbf{g}_m(\mathbf{e}))\big)^l\lambda(\mathcal{T}).$$

We have the canonical surjections

$$\mathrm{Sym}_R(E) \xrightarrow{\phi_1} \mathcal{R}_R(E) \xrightarrow{\phi_2} \mathcal{R}_R(I) \subset R[t].$$

Also, we can make the identification

$$\phi_2\Big(\phi_1\Big(\big((\mathbf{g}_1(\mathbf{e}))\cdots(\mathbf{g}_m(\mathbf{e}))\big)^l\Big)\Big) = \big((\mathbf{g}_1(\mathbf{f}t))\cdots(\mathbf{g}_m(\mathbf{f}t))\big)^l \in \mathcal{R}_R(I),$$

and from the birationality assumption we have that $\big((\mathbf{g}_1(\mathbf{f}t))\cdots(\mathbf{g}_m(\mathbf{f}t))\big)^l \neq 0$. Hence, it follows that

$$\phi_1\Big(\big((\mathbf{g}_1(\mathbf{e}))\cdots(\mathbf{g}_m(\mathbf{e}))\big)^l\Big)\phi_1\big(\lambda(\mathcal{T})\big) = 0 \in \mathcal{R}_R(E)$$

with $\phi_1\Big(\big((\mathbf{g}_1(\mathbf{e}))\cdots(\mathbf{g}_m(\mathbf{e}))\big)^l\Big) \neq 0$. Since $\mathcal{R}_R(E)$ is an integral domain, we get our claim $\phi_1\big(\lambda(\mathcal{T})\big) = 0$. □

3.4 Rational maps in the projective plane with saturated base ideal

In this section we focus on dominant rational maps $\mathcal{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose base ideal I is saturated and has dimension 1. To emphasize our interest in these cases, we recall, for instance, that the base ideal of birational maps of degree $d \leqslant 4$ must be saturated (see [67, Corollary 1.23]).

A straightforward application of the Auslander-Buchsbaum formula yields that the conditions of I being saturated and perfect are equivalent. Therefore, we will assume that I has a Hilbert-Burch presentation (see e.g. [47, Theorem 20.15]). We adopt Setup 3.31 with $r = 2$, and also the following.

Setup 3.46. *Assume that* $I = (f_0, f_1, f_2) \subset R(= \mathbb{k}[x_0, x_1, x_2])$ *is saturated and* $\dim(R/I) = 1$. *The presentation of* I *is given by*

$$0 \to R(-d-\mu_1) \oplus R(-d-\mu_2) \xrightarrow{\varphi} R(-d)^3 \to I \to 0, \tag{3.21}$$

where I *is generated by the maximal minors of* φ, $\mu_1 + \mu_2 = d$ *and* $\mu_1 \leqslant \mu_2$. *The matrix of* φ, *which we just denote by* φ, *is*

$$\varphi = \begin{pmatrix} a_{0,1} & a_{0,2} \\ a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

The main result of this section is Theorem 3.59 where we derive a very simple birationality criterion for rational maps $\mathcal{F}$ whose base ideal satisfy (3.21) with the additional assumption that

$\mu_1 = 1$. This result is based on a complete description of the equations of the Rees algebra of I in this setting, which is given in Theorem 3.57.

Before going further, we first notice that the degree of $\mathcal{F}$ under our assumptions is connected to the couple of integers (μ_1, μ_2) defined in (3.21).

Proposition 3.47. *Let $\mathcal{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a dominant rational map with a dimension 1 base ideal I that is saturated. Then,*

$$\deg(\mathcal{F}) \leqslant \mu_1\mu_2$$

with equality if and only if I is locally a complete intersection at its minimal primes.

Proof. The degree formula of Theorem 3.16 gives us $\deg(\mathcal{F}) = d^2 - e(\mathcal{B})$. We also know that $\deg(\mathcal{B}) \leqslant e(\mathcal{B})$ and $\deg(\mathcal{B}) = e(\mathcal{B})$ if and only if I is locally a complete intersection at its minimal primes. Now, using the resolution (3.21) and a simple computation with Hilbert polynomials , we get

$$\begin{aligned}\deg(\mathcal{B}) = P_{R/I}(t) &= \binom{t+2}{2} - 3\binom{t-d+2}{2} + \binom{t-d-\mu_1+2}{2} + \binom{t-d-\mu_2+2}{2} \\ &= d^2 - \mu_1\mu_2.\end{aligned}$$

Therefore, we deduce that $\deg(\mathcal{F}) \leqslant \mu_1\mu_2$ and $\deg(\mathcal{F}) = \mu_1\mu_2$ if and only if I is locally a complete intersection at its minimal primes. ☐

Properties of saturated base ideals

Below, we gather three technical results on some properties of the base ideal I under our assumptions. We will need them in the sequel.

Lemma 3.48. *Assume that $\dim(R/I) = 1$ and I is saturated. Then, the following statements hold:*

(i) $H^j_I(R) \neq 0$ *if and only if* $j = 2$.

(ii) $\mathrm{Ass}_R\left(H^2_I(R)\right)$ *is a finite set and equal to*

$$\mathrm{Ass}_R\left(H^2_I(R)\right) = \mathrm{Ass}_R\left(\mathrm{Ext}^2_R(R/I, R)\right) = \mathrm{Ass}_R\left(R/I\right).$$

Proof. (i) From the Grothendieck vanishing theorem [17, Theorem 6.1.2] we get that $H^j_I(R) = 0$ for $j > 3$. The connection of $\mathrm{grade}(I)$ with local cohomology [17, Theorem 6.2.7] implies that $H^j_I(R) = 0$ for $j < 2$ and $H^2_I(R) \neq 0$. Finally, a graded version of the Lichtenbaum-Hartshorne theorem [17, Theorem 14.1.16] yields $H^3_I(R) = 0$.

(ii) From [109, Proposition 1.1(b)] we have that $\mathrm{Ass}_R\left(H^2_I(R)\right) = \mathrm{Ass}_R\left(\mathrm{Ext}^2_R(R/I, R)\right)$. The module $\mathrm{Ext}^2_R(R/I, R)$ is the so-called canonical module $\omega_{R/I}$, and its associated primes are given by the unmixed part I^{un} of I (see [126, page 250, Lemma 1.9(c)]), that is

$$\mathrm{Ass}_R\left(\mathrm{Ext}^2_R(R/I, R)\right) = \left\{\mathfrak{p} \in \mathrm{Ass}_R(R/I) \mid \dim(R/\mathfrak{p}) = 1\right\}.$$

Since $\dim(R) = 3$, we finally get that I^{un} coincides with $I^{sat} = I$. $\square$

Using the present hypotheses we would like to better exploit the exact sequence

$$0 \to \mathcal{K} \to \mathrm{Sym}(I) \to \mathcal{R}(I) \to 0.$$

We recall that the symmetric algebra can be easily described with the presentation (3.21) of I, and its defining equations are given by

$$(g_1, g_2) = (y_0, y_1, y_2) \cdot \varphi. \tag{3.22}$$

Hence, we have an isomorphism

$$\mathrm{Sym}(I) \cong \mathfrak{A}/\left(g_1, g_2\right).$$

We also have that $\{g_1, g_2\}$ is a regular sequence in $\mathfrak{A}$ (see [133, Corollary 2.2]) and so the corresponding Koszul complex

$$\mathbb{L}_\bullet : \quad 0 \to \mathfrak{A}(-d, -2) \xrightarrow{\begin{pmatrix} -g_2 \\ g_1 \end{pmatrix}} \mathfrak{A}(-\mu_1, -1) \,\oplus\, \mathfrak{A}(-\mu_2, -1) \xrightarrow{(g_1, g_2)} \mathfrak{A} \tag{3.23}$$

is exact.

Lemma 3.49. *Assume that* $\dim(R/I) = 1$, *and* I *is saturated. Then, the torsion submodule* $\mathcal{K}$ *is described by the exact sequence*

$$0 \to \mathcal{K} \to H_I^2\left(\mathfrak{A}\right)(-d, -2) \xrightarrow{\begin{pmatrix} -g_2 \\ g_1 \end{pmatrix}} H_I^2\left(\mathfrak{A}\right)(-\mu_1, -1) \,\oplus\, H_I^2\left(\mathfrak{A}\right)(-\mu_2, -1).$$

Proof. We consider the double complex $\mathbb{L}_\bullet \otimes_R C_I^\bullet$. Computing with the second filtration we obtain the spectral sequence

$$^{II}E_2^{p,-q} = \begin{cases} H_I^p(\mathrm{Sym}(I)) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by using the first filtration we get that $^{I}E_1^{-p,q} = H_I^q(\mathbb{L}_p)$. Hence, from Lemma 3.48(i), the only row that does not vanish in $^{I}E_1^{\bullet,\bullet}$ is given by the complex

$$H_I^2(\mathbb{L}_\bullet) : \quad 0 \to H_I^2\left(\mathfrak{A}\right)(-d, -2) \to H_I^2\left(\mathfrak{A}\right)(-\mu_1, -1) \,\oplus\, H_I^2\left(\mathfrak{A}\right)(-\mu_2, -1) \to H_I^2\left(\mathfrak{A}\right) \to 0.$$

Thus we obtain

$$^{I}E_2^{-p,q} = \begin{cases} H_p\left(H_I^2(\mathbb{L}_\bullet)\right) & \text{if } q = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Since both spectral sequences collapse, from Lemma 1.10 we get

$$\mathcal{K} = \mathrm{H}_{\mathrm{I}}^{0}\big(\mathrm{Sym}(\mathrm{I})\big) \cong \mathrm{H}_2\big(\mathrm{H}_{\mathrm{I}}^{2}(\mathbb{L}_\bullet)\big),$$

and so the assertion follows. ☐

Notation 3.50. *For* $z = x_i$ *or* $z = y_j$ *and* $F \in \mathfrak{A}$, *we denote with* $\deg_z(F)$ *the maximal degree of the monomials of* F *in terms of* z.

Using the presentation matrix φ of I, we characterize when I is of linear type.

Lemma 3.51. *Assume that* $\dim(R/I) = 1$ *and* I *is saturated. Then,* I *is of linear type if and only if* $I_1(\varphi)$ *is an* $\mathfrak{m}$*-primary ideal.*

Proof. Using Setup 3.46, we have that $g_1 = a_{0,1}y_0 + a_{1,1}y_1 + a_{2,1}y_2$ and $g_2 = a_{0,2}y_0 + a_{1,2}y_1 + a_{2,2}y_2$.

($\Rightarrow$) Let us assume that $I_1(\varphi)$ is not $\mathfrak{m}$-primary. Then, we have that $I_1(\varphi) \supset I_2(\varphi) = I$ and $\mathrm{ht}(I_1(\varphi)) = \mathrm{ht}(I) = 2$. So the minimal primes of $I_1(\varphi)$ are contained in the set of associated primes of I. In particular, there exists some $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$ with $I_1(\varphi) \subset \mathfrak{p}$. From Lemma 3.48(ii) we have that $\mathfrak{p} \in \mathrm{Ass}_R\big(\mathrm{H}_I^2(R)\big)$, and this implies the existence of an element $0 \neq \nu \in \mathrm{H}_I^2(R)$ that is annihilated by $I_1(\varphi)$. Considering ν as an element in $\mathrm{H}_I^2(\mathfrak{A})$, we get $g_1 \cdot \nu = g_2 \cdot \nu = 0$. Therefore, from Lemma 3.49 we obtain $\mathcal{K} \neq 0$.

($\Leftarrow$) Here we suppose that $I_1(\varphi)$ is $\mathfrak{m}$-primary. By contradiction, we assume $\mathcal{K} \neq 0$, and choose $0 \neq w \in \mathcal{K}$. Since $\mathrm{H}_I^2(\mathfrak{A}) \cong \mathrm{H}_I^2(R) \otimes_{\mathbb{k}} S$, w can be written as $w = \sum_{i=1}^{l} \nu_i \otimes_{\mathbb{k}} m_i$ where $\nu_i \in \mathrm{H}_I^2(R)$ and m_i is a monomial in S. For each $0 \leqslant j \leqslant 2$, we have a unique decomposition

$$w = w_j + w_j^*,$$

where $w_j \neq 0$ is obtained by adding all the terms $\nu_i \otimes_{\mathbb{k}} m_i$ such that the value of $\deg_{y_j}(m_i)$ is maximal. From the condition $g_1 \cdot w = g_2 \cdot w = 0$, we automatically get that $a_{j,1} \cdot w_j = a_{j,2} \cdot w_j = 0$. Therefore, we have obtained that $I_1(\varphi)$ is composed of zero divisors in $\mathrm{H}_I^2(\mathfrak{A})$. By the isomorphism $\mathrm{H}_I^2(\mathfrak{A}) \cong \mathrm{H}_I^2(R) \otimes_{\mathbb{k}} S$ and Lemma 3.48(ii), we have that $\mathrm{Ass}_R\big(\mathrm{H}_I^2(\mathfrak{A})\big) = \mathrm{Ass}_R\big(\mathrm{H}_I^2(R)\big) = \mathrm{Ass}_R(R/I)$. Finally, the prime avoidance lemma implies that $I_1(\varphi) \subset \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$, and this contradicts the fact that I is saturated. ☐

Remark 3.52. *As in* [67]*, an alternative proof of Lemma 3.51 can be obtained from either* [71, Section 5] *or* [134, Proposition 3.7]. *Indeed, one can note that* I *is locally a complete intersection at its minimal primes if and only if* $I_1(\varphi)$ *is an* $\mathfrak{m}$*-primary ideal. Therefore, the result follows from the fact that* I *is an almost complete intersection.*

An effective birationality criterion in the case $\mu_1 = 1$

In this subsection, we focus on computing the defining equations of the Rees algebra in the case $\mu_1 = 1$ (Setup 3.46). As a corollary of this computation, we obtain a simple characterization of birationality in the particular case $\mu_1 = 1$ (Theorem 3.59) by means of the Jacobian dual criterion (see Section 3.3, but also [46]). Our proof is inspired by the method used in [39]. We shall see that it is enough to treat the following special case.

Setup 3.53. *Assume that* $\dim(R/I) = 1$ *and* I *is saturated. Suppose that the presentation matrix in (3.21) is of the form*

$$\varphi = \begin{pmatrix} x_0 & p_0 \\ -x_1 & p_1 \\ 0 & p_2 \end{pmatrix}.$$

Here we have that $g_1 = x_0y_0 - x_1y_1$ *and* $g_2 = p_0y_0 + p_1y_1 + p_2y_2$.

We now give a version of Lemma 3.49 that uses the more amenable ideal (x_0, x_1) as the support of the local cohomology modules.

Lemma 3.54. *Using Setup 3.53, the following statements hold:*

(i) $\mathcal{K} = H^0_{(x_0,x_1)}(\mathrm{Sym}(I))$.

(ii) The torsion submodule $\mathcal{K}$ *is determined by the following exact sequence*

$$0 \to \mathcal{K} \to H^2_{(x_0,x_1)}\big(\mathfrak{A}\big)(-d,-2) \xrightarrow{\begin{pmatrix} -g_2 \\ g_1 \end{pmatrix}} H^2_{(x_0,x_1)}\big(\mathfrak{A}\big)(-1,-1) \oplus H^2_{(x_0,x_1)}\big(\mathfrak{A}\big)(-d+1,-1).$$

(iii) Via this identification, we have that $\mathcal{K}$ *is generated by*

$$\mathcal{K} \;\cong\; \mathfrak{A} \cdot \Big\{ w_n \;\mid\; 0 \leqslant n \leqslant d-2 \;\; and \;\; g_2 \cdot w_n = 0 \Big\}$$

where each w_n *is of the form*

$$w_n = \sum_{i=0}^{d-2-n} \frac{1}{x_0^{i+1}x_1^{d-1-n-i}} y_0^{d-2-n-i} y_1^i \;\in\; \Big[H^2_{(x_0,x_1)}(\mathfrak{A})(0,-2)\Big]_{n-d}.$$

Proof. (i) From Lemma 3.49 we have that any $F \in \mathcal{K}$ can be written as $F = \sum_{i=1}^{l} a_i \mathbf{y}^{\gamma_i}$, where each $a_i \in H^2_I(R)$, and satisfies $g_1 \cdot F = g_2 \cdot F = 0$. Since $g_1 = x_0y_0 - x_1y_1$, we can conclude that there exists some $u > 0$ such that $x_0^u F = x_1^u F = 0$. From the fact that $I \subset (x_0, x_1)$, we get a neater description of $\mathcal{K}$ given by

$$\mathcal{K} = H^0_{(x_0,x_1)}(\mathcal{K}) = H^0_{(x_0,x_1)}\left(H^0_I(\mathrm{Sym}(I))\right) = H^0_{(x_0,x_1)}(\mathrm{Sym}(I)).$$

(ii) To obtain the required exact sequence we follow the same arguments as in the proof of Lemma 3.49. We consider the double complex $\mathbb{L}_\bullet \otimes_R C^\bullet_{(x_0,x_1)}$, where $\mathbb{L}_\bullet$ is the Koszul complex of (3.23). Examining the spectral sequences corresponding to the first and second filtrations of $\mathbb{L}_\bullet \otimes_R C^\bullet_{(x_0,x_1)}$, we obtain

$$\mathcal{K} = H^0_{(x_0,x_1)}(\mathrm{Sym}(I)) \cong H_2\big(H^2_{(x_0,x_1)}(\mathbb{L}_\bullet)\big).$$

From this isomorphism we get the claimed exact sequence.

(iii) First we note that $H^2_{(x_0,x_1)}(\mathfrak{A}) \cong \frac{1}{x_0x_1}\mathbb{k}[x_0^{-1}, x_1^{-1}, x_2, y_0, y_1, y_2]$. In this part, we describe a set of generators of the kernel of the multiplication map

$$H^2_{(x_0,x_1)}\big(\mathfrak{A}\big)(-d,-2) \xrightarrow{g_1} H^2_{(x_0,x_1)}\big(\mathfrak{A}\big)(-d+1,-1). \tag{3.24}$$

Using that $g_1 = x_0y_0 - x_1y_1$ does not depend on the variables x_2 and y_2, then a set of generators of the kernel of this map is given by just considering elements inside the subring $\frac{1}{x_0x_1}\mathbb{k}[x_0^{-1}, x_1^{-1}, y_0, y_1]$. Let $F \in \frac{1}{x_0x_1}\mathbb{k}[x_0^{-1}, x_1^{-1}, y_0, y_1]$, then we expand it as follows:

$$F = \sum_{i=l}^{m} F_i y_0^{\beta_i} y_1^i$$

where each $F_i \in \frac{1}{x_0x_1}\mathbb{k}[x_0^{-1}, x_1^{-1}]$. The condition $(x_0y_0 - x_1y_1)F = 0$ gives the relations

$$x_0F_l y_0^{\beta_l+1}y_1^l = 0, \quad x_1F_m y_0^{\beta_m}y_1^{m+1} = 0, \quad \text{and} \quad \left(x_0F_iy_0^{\beta_i+1} - x_1F_{i-1}y_0^{\beta_{i-1}}\right)y_1^i = 0$$

for $l+1 \leqslant i \leqslant m$.

We can easily conclude that a set of generators of the kernel of (3.24) is given by elements of the form

$$\frac{1}{x_0x_1^{m+1}}y_0^m + \frac{1}{x_0^2x_1^m}y_0^{m-1}y_1 + \cdots + \frac{1}{x_0^{m+1}x_1}y_1^m$$

where $m \geqslant 0$. Therefore, to conclude we only need to take into account the shifting of $-d$ in the grading part corresponding with R, and intersect with the elements that are also anihilated by the other equation g_2. $\square$

Now, we describe the process of computing the so-called Sylvester forms that have been successfully used in several papers like [39, 67, 78].

Algorithm 3.55. *Using Setup 3.53, we compute iteratively the set of forms $\mathcal{S}ylv_{(x_0,x_1)}(\varphi)$, as follows.*

(I) Set $i = 0$, $F_0 = g_2$ and $\mathcal{S}ylv_{(x_0,x_1)}(\varphi) = \emptyset$.

(II) While $F_i \in (x_0, x_1)$ we perform the following steps:

(a) *Write* F_i *in the convenient form* $F_i = (F_i)_{x_0} x_0 + (F_i)_{x_1} x_1$ *to get the equation*

$$\begin{pmatrix} g_1 \\ F_i \end{pmatrix} = \begin{pmatrix} y_0 & -y_1 \\ (F_i)_{x_0} & (F_i)_{x_1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

Then, the $(i+1)$*-th Sylvester form is computed with the determinant*

$$F_{i+1} = \det \begin{pmatrix} y_0 & -y_1 \\ (F_i)_{x_0} & (F_i)_{x_1} \end{pmatrix}.$$

(b) *Set* $\mathcal{S}ylv_{(x_0,x_1)}(\varphi) = \mathcal{S}ylv_{(x_0,x_1)}(\varphi) \cup \{F_{i+1}\}$.

(c) *Set* $i = i + 1$.

(III) *Set* $m = i$ *and return the set of computed forms* $\mathcal{S}ylv_{(x_0,x_1)}(\varphi) = \{F_1, \ldots, F_m\}$.

We emphasize for later use that $\mathrm{bideg}(F_i) = (d-1-i, i+1)$ *for each* $0 \leqslant i \leqslant m$.

The next lemma relates the torsion of the symmetric algebra with the Sylvester forms.

Lemma 3.56. *In Algorithm 3.55, for each* $1 \leqslant i \leqslant m$ *the following statements hold:*

(i) $\{g_1, F_i\}$ *is a regular sequence.*

(ii) *There is an isomorphism*

$$\left[\left(0 :_{\mathrm{Sym}(I)} (x_0, x_1)^i\right)\right]_{d-1-i} \cong \left[\frac{(g_1, F_i)}{(g_1)}\right]_{d-1-i}.$$

Proof. The proof is obtained by induction on i.

Let $i = 1$. Since $\{x_0, x_1\}$ and $\{g_1, g_2\}$ are regular sequences, from Wiebe's lemma (see e.g. [93, Proposition 3.8.1.6]) we get the following exact sequence

$$0 \to \mathfrak{A}/(x_0, x_1) \xrightarrow{F_1} \mathfrak{A}/(g_1, g_2) \xrightarrow{\left(\begin{smallmatrix} x_0 \\ x_1 \end{smallmatrix}\right)} [\mathfrak{A}/(g_1, g_2)]^2, \tag{3.25}$$

where F_1 is the first Sylvester form. Thus we have $F_1 \notin (g_1, g_2)$, and since g_1 is an irreducible polynomial, we get that $\{g_1, F_1\}$ is a regular sequence. From the fact that $\mathrm{bideg}(g_2) = (d-1, 1)$, for any $v \in \mathfrak{A}$ with $\deg_{\mathbf{x}}(v) \leqslant d-2$ the exact sequence (3.25) gives the following equivalences

$$v \in \left((g_1, g_2) : (x_0, x_1)\right) \Longleftrightarrow v \in (g_1, g_2, F_1) \Longleftrightarrow v \in (g_1, F_1).$$

In other words, we obtain the isomorphisms

$$\left[\left(0 :_{\mathrm{Sym}(I)} (x_0, x_1)\right)\right]_{d-2} \cong \left[\frac{(g_1, g_2, F_1)}{(g_1, g_2)}\right]_{d-2} \cong \left[\frac{(g_1, F_1)}{(g_1)}\right]_{d-2}.$$

Therefore, both conditions hold for $i = 1$.

Let $2 \leqslant i \leqslant m$. By induction we assume that conditions (i) and (ii) are satisfied for $i-1$. Again, from Weibe's lemma we get the exact sequence

$$0 \to \mathfrak{A}/(x_0, x_1) \xrightarrow{F_i} \mathfrak{A}/(g_1, F_{i-1}) \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}} [\mathfrak{A}/(g_1, F_{i-1})]^2, \tag{3.26}$$

where F_i is the i-th Sylvester form. By the same previous argument, it is clear that (g_1, F_i) is a regular sequence. Using the exactness of (3.26) and similar degree considerations, we have that

$$\nu \in \big((g_1, F_{i-1}) : (x_0, x_1)\big) \Longleftrightarrow \nu \in (g_1, F_{i-1}, F_i) \Longleftrightarrow \nu \in (g_1, F_i)$$

for any $\nu \in \mathfrak{A}$ with $\deg_{\mathbf{x}}(\nu) \leqslant d-1-i$. Thus, we also have the isomorphisms

$$\left[\frac{\big((g_1, F_{i-1}) : (x_0, x_1)\big)}{(g_1, F_{i-1})}\right]_{d-1-i} \cong \left[\frac{(g_1, F_{i-1}, F_i)}{(g_1, F_{i-1})}\right]_{d-1-i} \cong \left[\frac{(g_1, F_i)}{(g_1)}\right]_{d-1-i}.$$

Since $\deg_{\mathbf{x}}(F_{i-1}) = d-1-(i-1)$ and $[\mathrm{Sym}(I)]_{\leqslant d-2} \cong [\mathfrak{A}/(g_1)]_{\leqslant d-2}$, we get

$$\left[\frac{\big((g_1, F_{i-1}) : (x_0, x_1)\big)}{(g_1, F_{i-1})}\right]_{d-1-i} \cong \left[\left(\frac{(g_1, F_{i-1})}{(g_1)} :_{\mathrm{Sym}(I)} (x_0, x_1)\right)\right]_{d-1-i}.$$

From the inductive hypothesis we already have

$$\left[\left(0 :_{\mathrm{Sym}(I)} (x_0, x_1)^{i-1}\right)\right]_{d-1-(i-1)} \cong \left[\frac{(g_1, F_{i-1})}{(g_1)}\right]_{d-1-(i-1)}.$$

By assembling these isomorphisms we conclude that the condition (ii)

$$\left[\left(0 :_{\mathrm{Sym}(I)} (x_0, x_1)^{i}\right)\right]_{d-1-i} \cong \left[\frac{(g_1, F_i)}{(g_1)}\right]_{d-1-i}$$

also holds for the form F_i. Therefore, we have that both conditions are satisfied for all the Sylvester forms. $\square$

The following theorem gives explicit generators for the presentation of $\mathcal{R}(I)$. It can be seen as a natural generalization of both [39, Theorem 2.3] and [67, Theorem 2.7(i)].

Theorem 3.57. *Let $\mathcal{S}ylv_{(x_0,x_1)}(\varphi)$ be the set of Sylvester forms computed in Algorithm 3.55. Then, the defining equations of $\mathcal{R}(I)$ are minimally generated by*

$$\{g_1, g_2\} \cup \mathcal{S}ylv_{(x_0,x_1)}(\varphi).$$

In particular, it is minimally generated in the bi-degrees

$$(1,1),(d-1,1),(d-2,2),\ldots,(d-1-m,m+1).$$

Proof. Let $e = d-1-m$. In Lemma 3.56(ii) we proved that

$$\left[\left(0:_{\mathrm{Sym}(I)}(x_0,x_1)^m\right)\right]_e \cong \left[\frac{(g_1,F_m)}{(g_1)}\right]_e,$$

which implies that for any $j>0$ we have

$$\left[\left(0:_{\mathrm{Sym}(I)}(x_0,x_1)^{m+j}\right)\right]_{e-j} \cong \left[\left(\frac{(g_1,F_m)}{(g_1)}:_{\mathrm{Sym}(I)}(x_0,x_1)^j\right)\right]_{e-j}. \tag{3.27}$$

Since $F_m \notin (x_0,x_1)$ and $g_1 \in (x_0,x_1)$, we deduce that the term on the right is always equal to zero.

From Lemma 3.54(iii), a set of generators for $\mathcal{K}$ is given by elements of the form

$$w_n = \sum_{i=0}^{d-2-n} \frac{1}{x_0^{i+1}x_1^{d-1-n-i}} y_0^{d-2-n-i}y_1^i \in \left[H^2_{(x_0,x_1)}(\mathfrak{A})(0,-2)\right]_{n-d}.$$

Hence, for any $j>0$ we have that $(x_0,x_1)^{m+j}\cdot w_{e-j}=0$. The vanishing of the equation (3.27) implies that $w_{e-j}\notin\mathcal{K}$ for all $j>0$. Therefore, the elements $w_{d-2},w_{d-1},\ldots,w_e$ generate $\mathcal{K}$. Using the isomorphisms of Lemma 3.54(iii) and Lemma 3.56(ii), we identify w_{d-1-i} as a multiple of F_i, and this implies that $F_1,F_2,\ldots,F_m$ is also a set of generators of $\mathcal{K}$. Finally, simple degree considerations yield that $\{g_1,g_2,F_1,F_2,\ldots,F_m\}$ is a minimal set of generators. ☐

We are now ready to provide our birationality criterion. We notice that from Proposition 3.47, we have that the rational map $\mathcal{F}$ is birational for $d\leqslant 2$ under our assumptions. Therefore, we only need to consider the cases $d\geqslant 3$. Before stating the main result we make the following point.

Remark 3.58. *In the presentation matrix* φ *of (3.21), if* $\mu_1=1$ *and* $\mathrm{ht}(I_1(\varphi))=2$ *then the vector space spanned by the linear forms of the first column has dimension* 2*. Therefore, in this case we can make a linear change of coordinates and assume that* φ *is given as in Setup 3.53.*

The following result covers a family of birational maps that include the classical de Jonquières maps (see e.g. [67, §2.1]).

Theorem 3.59. *Let* $\mathcal{F}:\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a dominant rational map with a dimension* 1 *base ideal* I *that is saturated. Suppose that* φ *in (3.21) satisfies* $\mu_1=1$ *and* $d\geqslant 3$*. Then,* $\mathcal{F}$ *is birational if and only if the following conditions are satisfied:*

(i) $\mathrm{ht}(I_1(\varphi))=2$.

(ii) After the linear change of coordinates of Remark 3.58, in Algorithm 3.55 we have $m=d-2$.

Proof. After a linear change of coordinates the condition of birationality remains invariant. From the Jacobian dual criterion ([46, Theorem 2.18] or Theorem 3.39) we have that $\mathcal{F}$ is birational if and only if there is another equation of bi-degree $(1,*)$, and by Theorem 3.57 this is equivalent to $m = d - 2$. □

Chapter 4

Multiplicity of the saturated special fiber ring of height two perfect ideals

Let R be a polynomial ring and $I \subset R$ be a perfect ideal of height two minimally generated by forms of the same degree. In this chapter, we provide a formula for the multiplicity of the *saturated special fiber ring* of I. Interestingly, this formula is equal to an elementary symmetric polynomial in terms of the degrees of the syzygies of I. Applying ideas introduced in Chapter 3, we obtain the value of the j-multiplicity of I and an effective method for determining the degree and birationality of rational maps defined by homogeneous generators of I.

4.1 Multiplicity of the saturated special fiber ring

The following will be assumed in the rest of this chapter.

Setup 4.1. *Let $\mathbb{k}$ be a field, R be the polynomial ring $R = \mathbb{k}[x_0, x_1, \ldots, x_r]$, and $\mathfrak{m}$ be the maximal irrelevant ideal $\mathfrak{m} = (x_0, x_1, \ldots, x_r)$. Let I be a homogeneous ideal minimally generated by $I = (f_0, f_1, \ldots, f_s) \subset R$ where $\deg(f_i) = d$ and $s \geqslant r$. Let S be the polynomial ring $S = \mathbb{k}[y_0, y_1, \ldots, y_s]$, and $\mathfrak{A}$ be the bigraded polynomial ring $\mathfrak{A} = R \otimes_{\mathbb{k}} S = \mathbb{k}[x_0, \ldots, x_r, y_0, \ldots, y_s]$. Let Q be the special fiber ring $Q = \mathbb{k}[I_d] = \mathbb{k}[f_0, f_1, \ldots, f_s]$ of I.*

We assume that I is a perfect ideal of height two with Hilbert-Burch resolution of the form

$$0 \to \bigoplus_{i=1}^{s} R(-d-\mu_i) \xrightarrow{\varphi} R(-d)^{s+1} \to I \to 0. \tag{4.1}$$

We also suppose that I satisfies the condition G_{r+1}, that is

$$\mu(I_\mathfrak{p}) \leqslant \dim(R_\mathfrak{p}) \quad \textit{for all } \mathfrak{p} \in V(I) \subset \operatorname{Spec}(R) \textit{ such that } \operatorname{ht}(\mathfrak{p}) < r+1.$$

We shall determine the multiplicity of the following algebra.

Definition 4.2 (Definition 3.3). *The saturated special fiber ring of* I *is given by the algebra*

$$\widetilde{\mathfrak{F}_R(I)} = \bigoplus_{n=0}^{\infty} \left[\left(I^n : \mathfrak{m}^\infty\right)\right]_{nd}.$$

The Rees algebra $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n t^n \subset R[t]$ can be presented as a quotient of $\mathfrak{A}$ by using the map

$$\begin{array}{rcl} \Psi : \mathfrak{A} & \longrightarrow & \mathcal{R}(I) \subset R[t] \\ y_i & \mapsto & f_i t. \end{array}$$

We set $\mathrm{bideg}(x_i) = (1,0)$, $\mathrm{bideg}(y_j) = (0,1)$ and $\mathrm{bideg}(t) = (-d,1)$, which implies that Ψ is bihomogeneous of degree zero, and so $\mathcal{R}(I)$ has a structure of bigraded $\mathfrak{A}$-algebra. If M is a bigraded $\mathfrak{A}$-module and c a fixed integer, then we write

$$[M]_c = \bigoplus_{n \in \mathbb{Z}} M_{(c,n)}.$$

We remark that $[M]_c$ has a natural structure as a graded S-module.

As noted in Chapter 3, to study the algebra $\widetilde{\mathfrak{F}_R(I)}$ it is enough to consider the degree zero part in the R-grading of the bigraded $\mathfrak{A}$-module $H_{\mathfrak{m}}^1(\mathcal{R}(I))$ (see Lemma 2.2).

Remark 4.3. *Let* X *be the scheme* $X = \mathrm{Proj}_{R\text{-gr}}(\mathcal{R}(I))$, *where* $\mathcal{R}(I)$ *is only considered as a graded* R-*algebra. From* [47, Theorem A4.1], *we obtain the following short exact sequence*

$$0 \to [\mathcal{R}(I)]_0 \to H^0(X, \mathcal{O}_X) \to \left[H_{\mathfrak{m}}^1(\mathcal{R}(I))\right]_0 \to 0.$$

By identifying $Q \cong [\mathcal{R}(I)]_0$ *and* $\widetilde{\mathfrak{F}_R(I)} \cong H^0(X, \mathcal{O}_X)$, *we obtain the short exact sequence*

$$0 \to Q \to \widetilde{\mathfrak{F}_R(I)} \to \left[H_{\mathfrak{m}}^1(\mathcal{R}(I))\right]_0 \to 0. \tag{4.2}$$

Remark 4.4. *From Proposition 3.7*(i) *and Lemma 3.8*(ii) *we have that* $\widetilde{\mathfrak{F}_R(I)}$ *and* $\left[H_{\mathfrak{m}}^1(\mathcal{R}(I))\right]_0$ *have natural structures of finitely generated* Q-*modules.*

The Rees algebra is a very difficult object to study, but, under the present conditions, we have that the module $\left[H_{\mathfrak{m}}^1(\mathcal{R}(I))\right]_0$ coincides with $\left[H_{\mathfrak{m}}^1(\mathrm{Sym}(I))\right]_0$ (see Lemma 4.5(iii) below). So, the main idea is to consider the symmetric algebra instead of the Rees algebra. From the presentation (4.1) of I, we obtain the ideal

$$\mathcal{J} = (g_1, \ldots, g_s) = I_1\left([y_0, \ldots, y_s] \cdot \varphi\right)$$

of defining equations of the symmetric algebra. Thus, $\mathrm{Sym}(I)$ is a bigraded $\mathfrak{A}$-algebra presented by the quotient

$$\mathrm{Sym}(I) \cong \mathfrak{A}/\mathcal{J}.$$

We have the following canonical short exact sequence relating both algebras

$$0 \to \mathcal{K} \to \mathrm{Sym}(I) \to \mathcal{R}(I) \to 0, \tag{4.3}$$

where $\mathcal{K}$ is the R-torsion submodule of $\mathrm{Sym}(I)$.

We will consider the Koszul complex $\mathbb{L}_\bullet = K_\bullet(g_1, \ldots, g_s; \mathfrak{A})$ associated to $\{g_1, \ldots, g_s\}$:

$$\mathbb{L}_\bullet : \quad 0 \to \mathbb{L}_s \to \cdots \to \mathbb{L}_i \to \cdots \to \mathbb{L}_1 \to \mathbb{L}_0$$

where

$$\mathbb{L}_i = \bigwedge^i \left(\bigoplus_{j=1}^{s} \mathfrak{A}(-\mu_j, -1) \right). \tag{4.4}$$

This complex will not be exact in general, but the homology modules will have small enough Krull dimension. It will give us an "approximate resolution" of the symmetric algebra (see e.g. [101], [24]), from which we can read everything we need.

In the following lemma we gather some well-known properties of $\mathrm{Sym}(I)$ under the present conditions, we include them for the sake of completeness.

Lemma 4.5. *Using Setup 4.1, the following statements hold:*

(i) $\dim(\mathrm{Sym}(I)) = \max(\dim(R) + 1, \mu(I)) = \max(r + 2, s + 1)$.

(ii) $\mathcal{K} = H_m^0(\mathrm{Sym}(I))$.

(iii) $H_m^i(\mathcal{R}(I)) \cong H_m^i(\mathrm{Sym}(I))$ *for all* $i \geqslant 1$.

(iv) If $s \leqslant r + 1$, *then* $\mathrm{Sym}(I)$ *is a complete intersection.*

(v) For all $s \geqslant 1$, $\mathrm{Sym}(I)$ *is a complete intersection on the punctured spectrum of* R.

Proof. (i) It follows from Theorem 1.26 and the condition G_{r+1}.

(ii) It follows from Corollary 1.40 (also, see [104, §3.7]).

(iii) For each $i \geqslant 1$, the short exact sequence (4.3) yields the long exact sequence

$$H_m^i(\mathcal{K}) \to H_m^i(\mathrm{Sym}(I)) \to H_m^i(\mathcal{R}(I)) \to H_m^{i+1}(\mathcal{K}).$$

From part (ii) and [17, Corollary 2.1.7], we have that $H_m^i(\mathcal{K}) = H_m^{i+1}(\mathcal{K}) = 0$, and so we obtain the required isomorphism.

(iv) Using part (i), in this case we have that $\dim(\mathrm{Sym}(I)) = r + 2$. Hence, we get

$$\mathrm{ht}(\mathcal{J}) = \dim(\mathfrak{A}) - (r + 2) = (r + s + 2) - (r + 2) = s = \mu(\mathcal{J}),$$

and so $\mathrm{Sym}(I)$ is a complete intersection.

(v) For each $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{ht}(\mathfrak{p}) < r+1$, the same argument of part (i) now yields that $\dim(\mathrm{Sym}(I)_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}}) + 1$. Thus, we have

$$\mathrm{ht}(\mathcal{J}_{\mathfrak{p}}) = \dim(\mathfrak{A}_{\mathfrak{p}}) - \dim(\mathrm{Sym}(I)_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}}) + s + 1 - (\dim(R_{\mathfrak{p}}) + 1) = s = \mu(\mathcal{J}_{\mathfrak{p}}).$$

Then, for $i \geqslant 1$, the homology module $H_i(\mathbb{L}_\bullet)$ is supported on the maximal ideals of $\mathrm{Spec}(R)$, but since the associated primes $\mathrm{Ass}_R(H_i(\mathbb{L}_\bullet))$ are homogeneous, it necessarily gives that $\mathrm{Supp}_R(H_i(\mathbb{L}_\bullet)) = \{\mathfrak{m}\}$. Therefore, $\mathrm{Sym}(I)_{\mathfrak{p}}$ is a complete intersection for $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. □

The restriction to degree zero part in the R-grading of the equality $\mathcal{K} = H^0_{\mathfrak{m}}(\mathrm{Sym}(I))$ (Lemma 4.5(ii)) and the short exact sequence (4.3) yield the following

$$0 \to \left[H^0_{\mathfrak{m}}(\mathrm{Sym}(I))\right]_0 \to S \to Q \to 0, \tag{4.5}$$

under the identifications $[\mathrm{Sym}(I)]_0 = S$ and $[\mathcal{R}(I)]_0 = Q$.

The next proposition will be an important technical tool.

Proposition 4.6. *Assume Setup 4.1. Then, we have the following isomorphisms of bigraded $\mathfrak{A}$-modules*

$$H_i\left(H^{r+1}_{\mathfrak{m}}(\mathbb{L}_\bullet)\right) \cong \begin{cases} H^{r+1-i}_{\mathfrak{m}}(\mathrm{Sym}(I)) & \text{if } i \leqslant r+1 \\ H_{i-r-1}(\mathbb{L}_\bullet) & \text{if } i \geqslant r+2, \end{cases}$$

where $H^{r+1}_{\mathfrak{m}}(\mathbb{L}_\bullet)$ represents the complex obtained after applying the functor $H^{r+1}_{\mathfrak{m}}(\bullet)$ to $\mathbb{L}_\bullet$.

Proof. Let $\mathbb{G}^{\bullet,\bullet}$ be the double complex $\mathbb{G}^{\bullet,\bullet} = \mathbb{L}_\bullet \otimes_R C^\bullet_{\mathfrak{m}}$, where $C^\bullet_{\mathfrak{m}}$ is the Čech complex corresponding with the maximal irrelevant ideal $\mathfrak{m}$.

Since we have that

$$H^p_{\mathfrak{m}}(\mathfrak{A}) \cong \begin{cases} \frac{1}{x_0x_1\cdots x_r}\mathbb{k}[x_0^{-1}, x_1^{-1}, \ldots, x_r^{-1}] \otimes_{\mathbb{k}} S & \text{if } p = r+1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

the spectral sequence coming from the first filtration is given by

$${}^{\mathrm{I}}E_1^{-p,q} = \begin{cases} H^{r+1}_{\mathfrak{m}}(\mathbb{L}_p) & \text{if } q = r+1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, Lemma 4.5(v) implies that $(\mathbb{L}_\bullet)_{\mathfrak{p}}$ is exact for all $\mathfrak{p} \in \mathrm{Spec}(R)\setminus\{\mathfrak{m}\}$. So, for all $i \geqslant 1$, $H_i(\mathbb{L}_\bullet)$ is supported on $V(\mathfrak{m})$ and the Grothendieck vanishing theorem (see e.g. [17, Theorem 6.1.2]) implies that

$$H^j_{\mathfrak{m}}(H_i(\mathbb{L}_\bullet)) = 0$$

for all $j \geqslant 1$. Also, we have that

$$H^0_{\mathfrak{m}}(H_i(\mathbb{L}_\bullet)) = H_i(\mathbb{L}_\bullet)$$

for $i \geqslant 1$. Therefore, the spectral sequence corresponding with the second filtration is given by

$$ {}^{\mathrm{II}}E_2^{p,-q} \cong \begin{cases} H_{\mathfrak{m}}^p(\mathrm{Sym}(I)) & \text{if } q = 0 \\ H_q(\mathbb{L}_\bullet) & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases} $$

Finally, from the convergence of both spectral sequences we obtain the following isomorphisms of bigraded $\mathfrak{A}$-modules

$$ H_i\left(H_{\mathfrak{m}}^{r+1}(\mathbb{L}_\bullet)\right) \cong H^{r+1-i}\left(\mathrm{Tot}(\mathbb{G}^{\bullet,\bullet})\right) \cong \begin{cases} H_{\mathfrak{m}}^{r+1-i}(\mathrm{Sym}(I)) & \text{if } i \leqslant r+1 \\ H_{i-r-1}(\mathbb{L}_\bullet) & \text{if } i \geqslant r+2 \end{cases} $$

for all $i \geqslant 0$. □

The following lemma contains some dimension computations that will be needed in the proof of Theorem 4.8. The first one shows that I has maximal analytic spread and it is obtained directly from [142]. The second one is a curious interplay between the algebraic properties of I and the geometric features of the corresponding rational map (4.15), that follows from Proposition 3.14.

Lemma 4.7. *Using Setup 4.1, the following statements hold:*

(i) $\ell(I) = \dim(Q) = r + 1$.

(ii) $\dim\left(\left[H_{\mathfrak{m}}^i(\mathrm{Sym}(I))\right]_0\right) \leqslant r$ *for all* $i \geqslant 2$.

Proof. (i) In the case $r = s$, we have $\mu(I) = s + 1 = r + 1 = \dim(R)$, and so the condition G_{r+1} is equivalent to G_∞. Thus, when $r = s$, we get from Theorem 1.37 that I is of linear type and so $\dim(Q) = r + 1$. When $s \geqslant r + 1$, then the result follows from Proposition 1.41.

(ii) Let $i \geqslant 2$. From Lemma 4.5(iii), we have $\left[H_{\mathfrak{m}}^i(\mathrm{Sym}(I))\right]_0 \cong \left[H_{\mathfrak{m}}^i(\mathcal{R}(I))\right]_0$. Since I has maximal analytic spread $\ell(I) = \dim(Q) = r + 1$, then the corresponding rational map is generically finite and so the inequality follows directly from Proposition 3.14. □

Now we are ready for the proof of the main theorem of this chapter.

Theorem 4.8. *Let* $I \subset R = \mathbb{k}[x_0, x_1, \ldots, x_r]$ *be a homogeneous ideal minimally generated by* $s + 1$ *forms* $\{f_0, f_1, \ldots, f_s\}$ *of the same degree* d*, where* $s \geqslant r$*. Suppose the following two conditions:*

(i) I *is perfect of height two with Hilbert-Burch resolution of the form*

$$ 0 \to \bigoplus_{i=1}^{s} R(-d-\mu_i) \xrightarrow{\varphi} R(-d)^{s+1} \to I \to 0. $$

(ii) I *satisfies the condition* G_{r+1}.

Then, the multiplicity of the saturated special fiber ring $\widetilde{\mathfrak{F}_R(I)}$ *is given by*

$$e\left(\widetilde{\mathfrak{F}_R(I)}\right) = e_r(\mu_1, \mu_2, \ldots, \mu_s),$$

where $e_r(\mu_1, \mu_2, \ldots, \mu_s)$ *represents the* r*-th elementary symmetric polynomial*

$$e_r(\mu_1, \mu_2, \ldots, \mu_s) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant s} \mu_{j_1}\mu_{j_2}\cdots\mu_{j_r}.$$

Proof. We analyze the homology modules of the complex

$$\mathbb{F}_\bullet = \left[H_m^{r+1}(\mathbb{L}_\bullet)\right]_0 : \quad 0 \to \left[H_m^{r+1}(\mathbb{L}_s)\right]_0 \to \cdots \to \left[H_m^{r+1}(\mathbb{L}_1)\right]_0 \to \left[H_m^{r+1}(\mathbb{L}_0)\right]_0$$

obtained by applying $H_m^{r+1}(\bullet)$ to the complex $\mathbb{L}_\bullet$ and then restricting to the degree zero part in the R-grading. From (4.4) and (4.6), we can make the identification

$$\mathbb{F}_i = \left[H_m^{r+1}(\mathbb{L}_i)\right]_0 \cong S(-i)^{m_i},$$

where

$$m_i = \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant s} \binom{\sum_{e=1}^{i} \mu_{j_e} - 1}{r}.$$

First, from Proposition 4.6 we have

$$H_i(\mathbb{F}_\bullet) \cong [H_{i-r-1}(\mathbb{L}_\bullet)]_0 \text{ for } i \geqslant r+2,$$

then the fact that $[\mathbb{L}_k]_0 = 0$ for $k \geqslant 1$ (see (4.4)) yields the vanishing

$$H_i(\mathbb{F}_\bullet) = 0 \text{ for all } i \geqslant r+2. \tag{4.7}$$

On the other hand, Proposition 4.6 also gives that

$$H_i(\mathbb{F}_\bullet) \cong \left[H_m^{r+1-i}(\mathrm{Sym}(I))\right]_0 \quad \text{for } i \leqslant r+1,$$

and Lemma 4.7(ii) implies that

$$\dim\left(H_i(\mathbb{F}_\bullet)\right) \leqslant r \text{ for all } i \leqslant r-1. \tag{4.8}$$

Let $B_\bullet$, $Z_\bullet$ and $H_\bullet$ be the boundaries, cycles and homology modules of the complex $\mathbb{F}_\bullet$, respectively. We have the following short exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0$$
$$0 \to Z_i \to \mathbb{F}_i \to B_{i-1} \to 0$$

for all i. By using the additivity of Hilbert series and assembling all these short exact sequences we obtain the following equation

$$\sum_{i=0}^{s}(-1)^i H_{H_i}(T) = \sum_{i=0}^{s}(-1)^i H_{\mathbb{F}_i}(T).$$

Using (4.7) and (4.8), it follows that $H_{H_i}(T) = 0$ for $i \geqslant r+2$, and that we can write

$$H_{H_i}(T) = \frac{G_i(T)}{(1-T)^{e_i}} \quad \text{for } i \leqslant r-1$$

where $G_i(T) \in \mathbb{Z}[T]$ and $e_i = \dim(H_i) \leqslant r$ (see e.g. [19, Section 4.1]). Therefore, we obtain the following equation

$$\frac{C(T)}{(1-T)^{s+1}} + (-1)^r H_{H_r}(T) + (-1)^{r+1} H_{H_{r+1}}(T) = \frac{G(T)}{(1-T)^{s+1}}$$

where

$$C(T) = \sum_{i=0}^{r-1}(-1)^i(1-T)^{s+1-e_i}G_i(T) \quad \text{and} \quad G(T) = \sum_{i=0}^{s}(-1)^i m_i T^i.$$

The isomorphisms of Proposition 4.6 yield that

$$H_{\left[H_{\mathfrak{m}}^1(\mathrm{Sym}(I))\right]_0}(T) = H_{\left[H_{\mathfrak{m}}^0(\mathrm{Sym}(I))\right]_0}(T) + \frac{(-1)^r G(T) + (-1)^{r+1} C(T)}{(1-T)^{s+1}} \tag{4.9}$$

From the short exact sequence (4.5) we obtain that

$$H_{\left[H_{\mathfrak{m}}^0(\mathrm{Sym}(I))\right]_0}(T) = H_S(T) - H_Q(T) = \frac{1}{(1-T)^{s+1}} - H_Q(T), \tag{4.10}$$

and the short exact sequence (4.2) and Lemma 4.5(iii) yield that

$$H_{\widetilde{\mathfrak{F}_R(I)}}(T) = H_Q(T) + H_{\left[H_{\mathfrak{m}}^1(\mathrm{Sym}(I))\right]_0}(T). \tag{4.11}$$

Hence, by summing up (4.9), (4.10) and (4.11) we get

$$H_{\widetilde{\mathfrak{F}_R(I)}}(T) = \frac{1 + (-1)^r G(T) + (-1)^{r+1} C(T)}{(1-T)^{s+1}}.$$

Let $F(T) = 1 + (-1)^r G(T) + (-1)^{r+1} C(T)$. From Lemma 4.7$(i)$ we have that $\dim\left(\widetilde{\mathfrak{F}_R(I)}\right) =$

$\dim(Q) = r+1$, then well-known properties of Hilbert series (see e.g. [19, Section 4.1]) give us

$$F(T) = (1-T)^{s-r}F_1(T),$$

where $F_1(1) \neq 0$ and $e\left(\widetilde{\mathfrak{F}_R(I)}\right) = F_1(1)$. The fact that $e_i \leqslant r$ for $i \leqslant r-1$, implies that $C^{(s-r)}(1) = 0$. By denoting

$$P(T) = 1 + (-1)^r G(T) = 1 + \sum_{i=0}^{s} (-1)^{r+i} m_i T^i,$$

we get $P^{(s-r)}(1) = F^{(s-r)}(1)$, and so by taking the $(s-r)$-th derivatives of $F(T)$ and $P(T)$ we obtain that

$$\begin{aligned}(-1)^{s-r}(s-r)! \cdot F_1(1) &= P^{(s-r)}(1) \\ &= \begin{cases} 1 + \sum_{i=0}^{r} (-1)^{r+i} m_i & \text{if } s = r \\ \sum_{i=s-r}^{s} (-1)^{r+i} m_i (s-r)! \binom{i}{s-r} & \text{if } s > r. \end{cases}\end{aligned}$$

The substitution of $e\left(\widetilde{\mathfrak{F}_R(I)}\right) = F_1(1)$ gives us that

$$e\left(\widetilde{\mathfrak{F}_R(I)}\right) = \begin{cases} 1 + \sum_{i=0}^{r} (-1)^{r+i} m_i & \text{if } s = r \\ \sum_{i=s-r}^{s} (-1)^{s+i} m_i \binom{i}{s-r} & \text{if } s > r. \end{cases} \tag{4.12}$$

Finally, the result is obtained from Lemma 4.9(iii), (iv) below. □

In the following lemma we use simple combinatorial techniques to reduce the equation (4.12).

Lemma 4.9. *The following formulas hold:*

(i) For $0 \leqslant k \leqslant r$,

$$\sum_{i=\max\{k,s-r\}}^{s} (-1)^i \binom{i}{s-r}\binom{s-k}{i-k} = \begin{cases} (-1)^s & \text{if } k = r \\ 0 & \text{if } k < r. \end{cases}$$

(ii) For $1 \leqslant \ell \leqslant r$,

$$\sum_{i=s-r}^{s} (-1)^i \binom{i}{s-r} \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant s} \Big(\sum_{e=1}^{i} \mu_{j_e}\Big)^{\ell} = \begin{cases} (-1)^s r! \cdot e_r(\mu_1, \ldots, \mu_s) & \text{if } \ell = r \\ 0 & \text{if } \ell < r. \end{cases}$$

(iii) For $s > r$,

$$\sum_{i=s-r}^{s} (-1)^i \binom{i}{s-r} \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant s} \binom{\sum_{e=1}^{i} \mu_{j_e} - 1}{r} = (-1)^s \cdot e_r(\mu_1, \dots, \mu_s).$$

(iv) For $s = r$,

$$1 + \sum_{i=0}^{r} (-1)^{i+r} \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant r} \binom{\sum_{e=1}^{i} \mu_{j_e} - 1}{r} = \mu_1 \mu_2 \cdots \mu_r.$$

Proof. (i) We start from the identity

$$(1-T)^{s-k} T^k = \sum_{i=k}^{s} (-1)^{i-k} \binom{s-k}{i-k} T^i,$$

then by taking the $(s-r)$-th derivative in both sides we get

$$\left((1-T)^{s-k} T^k\right)^{(s-r)} = \sum_{i=\max\{k, s-r\}}^{s} (-1)^{i-k} \binom{s-k}{i-k} (s-r)! \binom{i}{s-r} T^{i-s+r}.$$

Since $s-k \geqslant s-r$, the substitution $T = 1$ yields the result.

(ii) For each set of indexes $\{j_1, \dots, j_i\}$ we have

$$\left(\sum_{e=1}^{i} \mu_{j_e}\right)^{\ell} = \sum_{\ell_1 + \cdots + \ell_i = \ell} \binom{\ell}{\ell_1, \dots, \ell_i} \mu_{j_1}^{\ell_1} \cdots \mu_{j_i}^{\ell_i}. \tag{4.13}$$

We will proceed by determining the coefficients of each of the monomials $\mu_{j_1}^{\ell_1} \cdots \mu_{j_i}^{\ell_i}$ in the equation. Since $\binom{\ell}{\ell_1, \dots, \ell_i} = \binom{\ell}{\ell_1, \dots, \ell_i, 0}$, we can consider the case where $\ell_1 \neq 0, \dots, \ell_k \neq 0$.

We fix $1 \leqslant k \leqslant r$ and the monomial $\mu_{i_1}^{b_1} \cdots \mu_{i_k}^{b_k}$ where $b_1 \neq 0, \dots, b_k \neq 0$ and $b_1 + \cdots + b_k = \ell$. For each set of indexes $\{j_1, \dots, j_i\} \supset \{i_1, \dots, i_k\}$, the monomial $\mu_{i_1}^{b_1} \cdots \mu_{i_k}^{b_k}$ appears once in the equation (4.13), and the number of these sets is equal to $\binom{s-k}{i-k}$. Thus, for each $i \geqslant k$, the coefficient of $\mu_{i_1}^{b_1} \cdots \mu_{i_k}^{b_k}$ in the expression

$$\sum_{1 \leqslant j_1 < \cdots < j_i \leqslant s} \left(\sum_{e=1}^{i} \mu_{j_e}\right)^{\ell}$$

is equal to $\binom{s-k}{i-k}\binom{\ell}{b_1,\ldots,b_k}$. So, the total coefficient of $\mu_{i_1}^{b_1}\cdots\mu_{i_k}^{b_k}$ is given by

$$\binom{\ell}{b_1,\ldots,b_k}\sum_{i=\max\{k,s-r\}}^{s}(-1)^i\binom{i}{s-r}\binom{s-k}{i-k}.$$

From part (i), we have that this coefficient vanishes when $k<r$ and that it is equal to $(-1)^s r!$ when $k=r$ because $\ell\leqslant r$.

Therefore, for $\ell<r$ we have that the equation vanishes, and for $\ell=r$ that the only monomials in the equation are those of the elementary symmetric polynomial $e_r(\mu_1,\ldots,\mu_s)$ and the coefficient of all of them is $(-1)^s r!$.

(iii) We can write

$$\begin{aligned}\binom{\sum_{e=1}^{i}\mu_{j_e}-1}{r} &= \frac{\left(\sum_{e=1}^{i}\mu_{j_e}-1\right)\left(\sum_{e=1}^{i}\mu_{j_e}-2\right)\cdots\left(\sum_{e=1}^{i}\mu_{j_e}-r\right)}{r!} \\ &= \frac{1}{r!}\sum_{\ell=0}^{r}(-1)^{r-\ell}e_{r-\ell}(1,2,\ldots,r)\left(\sum_{e=1}^{i}\mu_{j_e}\right)^{\ell}.\end{aligned}\tag{4.14}$$

Therefore, by summing up and using part (ii), we obtain the required formula.

(iv) From equation (4.14) and part (ii) we have

$$\sum_{i=0}^{r}(-1)^{i+r}\sum_{1\leqslant j_1<\cdots<j_i\leqslant r}\binom{\sum_{e=1}^{i}\mu_{j_e}-1}{r}=\mu_1\mu_2\cdots\mu_r+\sum_{i=1}^{r}(-1)^i\binom{r}{i}.$$

Thus we get the result from the identity $\sum_{i=0}^{r}(-1)^i\binom{r}{i}=0$. □

From Theorem 4.8 we obtain a closed formula for the j-multiplicity of I.

Corollary 4.10. *Assume all the hypotheses and notations of Theorem 4.8. Then, the j-multiplicity of I is given by*

$$j(I)=d\cdot e_r(\mu_1,\mu_2,\ldots,\mu_s).$$

Proof. From Lemma 3.10 we have that $j(I)=d\cdot e\left(\widetilde{\mathfrak{F}_R(I)}\right)$, then the result follows from the computation of Theorem 4.8. □

4.2 Degree of rational maps

In this short section we study the degree of the rational map

$$\begin{aligned} \mathcal{F} : \mathbb{P}^r &\dashrightarrow \mathbb{P}^s \\ (x_0 : \cdots : x_r) &\mapsto \big(f_0(x_0, \ldots, x_r) : \cdots : f_s(x_0, \ldots, x_r)\big), \end{aligned} \tag{4.15}$$

whose base base ideal $I = (f_0, f_1, \ldots, f_s)$ satisfies all the conditions of Setup 4.1. Here we obtain a suitable generalization of [102, Theorem 4.9 $(1), (2)$], where we relate the degree of $\mathcal{F}$ and the degree of its image with the formula obtained in Theorem 4.8. An interesting result is that $\mathcal{F}$ is birational onto its image if and only if the degree of the image is the maximum possible.

Let $Y \subset \mathbb{P}^s$ be the closure of the image of $\mathcal{F}$. From Lemma 4.7(i) we have that $\dim(Y) = \dim(\mathbb{P}^r)$, and that the degree of $\mathcal{F}$ is equal to the dimension of the field extension

$$\deg(\mathcal{F}) = [K(\mathbb{P}^r) : K(Y)]\,,$$

where $K(\mathbb{P}^r)$ and $K(Y)$ represent the fields of rational functions of $\mathbb{P}^r$ and Y, respectively.

The main result of this section is a simple corollary of Theorem 3.4 and Theorem 4.8.

Corollary 4.11. *Assume all the hypotheses and notations of Theorem 4.8. Let $\mathcal{F}$ be the rational map $\mathcal{F} : \mathbb{P}^r \dashrightarrow \mathbb{P}^s$ given by*

$$(x_0 : \cdots : x_r) \mapsto \big(f_0(x_0, \ldots, x_r) : \cdots : f_s(x_0, \ldots, x_r)\big),$$

and $Y \subset \mathbb{P}^s$ be the closure of the image of $\mathcal{F}$. Then, the following two statements hold:

(i) $\deg(\mathcal{F}) \cdot \deg_{\mathbb{P}^s}(Y) = e_r(\mu_1, \mu_2, \ldots, \mu_s)$.

(ii) $\mathcal{F}$ is birational onto its image if and only if $\deg_{\mathbb{P}^s}(Y) = e_r(\mu_1, \mu_2, \ldots, \mu_s)$.

Proof. From Theorem 3.4(iii) we have that $e\left(\widetilde{\mathfrak{F}_R(I)}\right) = \deg(\mathcal{F}) \cdot \deg_{\mathbb{P}^s}(Y)$, then the result is obtained from the computation of Theorem 4.8. □

We have that in the literature special cases of Corollary 4.11 have appeared before. For instance, in [40, Proposition 5.3] a particular case of Corollary 4.11 was obtained for parameterized surfaces. In the following simple corollaries, we proof the same result of [102, Theorem 4.9 $(1), (2)$], and we generalize [22, Proposition 5.2].

Corollary 4.12. *With the same notations above, if $r = 1$, i.e. $\mathcal{F}$ is of the form $\mathcal{F} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^s$, then* $\deg(\mathcal{F}) \cdot \deg_{\mathbb{P}^s}(Y) = d$.

Proof. We only need to note that $e_1(\mu_1, \mu_2, \ldots, \mu_s) = d$. □

Corollary 4.13. *With the same notations above, if $r = s$, i.e. $\mathcal{F}$ is of the form $\mathcal{F} : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$, then* $\deg(\mathcal{F}) = \mu_1 \mu_2 \cdots \mu_r$.

Proof. In this case we have $Y = \mathbb{P}^r$ and so $\deg_{\mathbb{P}^r}(Y) = 1$. Hence the equality follows from the fact that $e_r(\mu_1, \mu_2, \dots, \mu_r) = \mu_1 \mu_2 \cdots \mu_r$. $\square$

Chapter 5

Degree of rational maps via specialization

In this chapter, we consider the behavior of the degree of a rational map under specialization of the coefficients of the defining linear system. The method rests on the classical idea of Kronecker as applied to the context of projective schemes and their specializations. For the theory to work we are led to develop the details of rational maps and their graphs when the ground ring of coefficients is a Noetherian integral domain.

Note. The results of this chapter are based on joint work with Aron Simis.

5.1 Terminology and notation

Let R be a Noetherian ring and $I \subset R$ be an ideal. We recall the following definitions and notions from Section 1.1.

Definition 5.1. *The Rees algebra of* I *is defined as the* R*-subalgebra*

$$\mathcal{R}_R(I) := R[It] = \bigoplus_{n \geqslant 0} I^n t^n \subset R[t],$$

and the associated graded ring of I *is given by*

$$\mathrm{gr}_I(R) := \mathcal{R}_R(I)/I\mathcal{R}_R(I) \cong \bigoplus_{n \geqslant 0} I^n/I^{n+1}.$$

If, moreover, R *is local, with maximal ideal* $\mathfrak{m}$*, we define the fiber cone of* I *to be*

$$\mathfrak{F}_R(I) := \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I) \cong \mathrm{gr}_I(R)/\mathfrak{m}\mathrm{gr}_I(R),$$

and the analytic spread of I*, denoted by* $\ell(I)$*, to be the (Krull) dimension of* $\mathfrak{F}_R(I)$.

The following notation will prevail throughout most of the chapter.

Setup 5.2. *Let A be a Noetherian ring of finite Krull dimension. Let $(R, \mathfrak{m})$ denote a standard graded algebra over $A = [R]_0$ and $\mathfrak{m}$ be its graded irrelevant ideal $\mathfrak{m} = ([R]_1)$. Let $S := A[y_0, \ldots, y_s]$ denote a standard graded polynomial ring over A.*

Let $I \subset R$ be a homogeneous ideal generated by $s+1$ polynomials $\{f_0, \ldots, f_s\} \subset R$ of the same degree $d > 0$ – in particular, $I = ([I]_d)$. Consider the bigraded A-algebra

$$\mathfrak{A} := R \otimes_A S = R[y_0, \ldots, y_s],$$

where $\mathrm{bideg}([R]_1) = (1, 0)$ and $\mathrm{bideg}(y_i) = (0, 1)$. By setting $\mathrm{bideg}(t) = (-d, 1)$, then $\mathcal{R}_R(I) = R[It]$ inherits a bigraded structure over A. We have a bihomogeneous (of degree zero) R-homomorphism

$$\mathfrak{A} \longrightarrow \mathcal{R}_R(I) \subset R[t], \quad y_i \mapsto f_i t. \tag{5.1}$$

Thus, the bigraded structure of $\mathcal{R}_R(I)$ is given by

$$\mathcal{R}_R(I) = \bigoplus_{c,n \in \mathbb{Z}} [\mathcal{R}_R(I)]_{c,n} \quad \text{and} \quad [\mathcal{R}_R(I)]_{c,n} = [I^n]_{c+nd} t^n.$$

We are primarily interested in the R-grading of the Rees algebra, namely, $[\mathcal{R}_R(I)]_c = \bigoplus_{n=0}^{\infty} [\mathcal{R}_R(I)]_{c,n}$, and of particular interest is

$$[\mathcal{R}_R(I)]_0 = \bigoplus_{n=0}^{\infty} [I^n]_{nd} t^n = A\left[[I]_d t\right] \cong A\left[[I]_d\right] = \bigoplus_{n=0}^{\infty} [I^n]_{nd} \subset R.$$

Clearly, $\mathcal{R}_R(I) = [\mathcal{R}_R(I)]_0 \oplus \left(\bigoplus_{c \geqslant 1} [\mathcal{R}_R(I)]_c\right) = [\mathcal{R}_R(I)]_0 \oplus \mathfrak{m}\mathcal{R}_R(I)$. Therefore, we get

$$A\left[[I]_d\right] \cong [\mathcal{R}_R(I)]_0 \cong \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I)$$

as graded A-algebras.

Definition 5.3. *Because of its resemblance to the fiber cone in the case of a local ring, we refer to the right-most algebra above as the (relative) fiber cone of I, and often identify it with the A-subalgebra $A\left[[I]_d\right] \subset R$ by the above natural isomorphism. It will also be denoted by $\mathfrak{F}_R(I)$.*

Remark 5.4 (Definition 1.4)**.** *If R has a distinguished or special maximal ideal $\mathfrak{m}$ (that is, if R is graded with graded irrelevant ideal $\mathfrak{m}$ or if R is local with maximal ideal $\mathfrak{m}$), then the fiber cone also receives the name of special fiber ring.*

5.2 Rational maps over an integral domain

In this part we develop the main points of the theory of rational maps with source and target projective varieties defined over an arbitrary Noetherian integral domain of finite Krull dimension. Similar results will take place in the case the source is a biprojective (more generally, a multi-projective) variety – the interested reader can readily provide the main analogous results. From now on assume that R is an integral domain, which in particular implies that $A = [R]_0$ is also an integral domain. Some of the subsequent results will also work assuming that R is reduced, but additional technology would be required.

Dimension

In this subsection we consider a simple way of constructing chains of relevant graded prime ideals and draw upon it to algebraically describe the dimension of projective schemes. These results are probably well-known, but we include them anyway for the sake of completeness.

For convenience of the reader we recall the following easy fact.

Lemma 5.5. *Let B be a commutative ring and $A \subset B$ a subring. Then, for any minimal prime $\mathfrak{p} \in \operatorname{Spec}(A)$ there exists a minimal prime $\mathfrak{P} \in \operatorname{Spec}(B)$ such that $\mathfrak{p} = \mathfrak{P} \cap A$.*

Proof. First, there is some prime of B lying over $\mathfrak{p}$. Indeed, any prime ideal of the ring of fractions $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$ is the image of a prime ideal $P \subset B$ not meeting $A \setminus \mathfrak{p}$, hence contracting to $\mathfrak{p}$.

For any descending chain of prime ideals $P = P_0 \supsetneq P_1 \supsetneq \cdots$ such that $P_i \cap A \subseteq \mathfrak{p}$ for every i, their intersection Q is prime and obviously $Q \cap A \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is minimal, then $Q \cap A = \mathfrak{p}$.

Therefore, Zorn's lemma yields the existence of a minimal prime in B contracting to $\mathfrak{p}$. $\square$

Proposition 5.6. *Let A be a Noetherian integral domain of finite Krull dimension $k = \dim(A)$ and let R denote a finitely generated graded algebra over A with $[R]_0 = A$. Let $\mathfrak{m} := (R_+)$ be the graded irrelevant ideal of R. If $\operatorname{ht}(\mathfrak{m}) \geqslant 1$, then there exists a chain of graded prime ideals*

$$0 = \mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_{k-1} \subsetneq \mathfrak{P}_k$$

such that $\mathfrak{P}_k \not\supseteq \mathfrak{m}$.

Proof. Proceed by induction on $k = \dim(A)$.

The case $k = 0$ it is clear or vacuous. Thus, assume that $k > 0$.

Let $\mathfrak{n}$ be a maximal ideal of A with $\operatorname{ht}(\mathfrak{n}) = k$. By [110, Theorem 13.6] we can choose $0 \neq a \in \mathfrak{n} \subset A$ such that $\operatorname{ht}(\mathfrak{n}/aA) = \operatorname{ht}(\mathfrak{n}) - 1$. Let $\mathfrak{q}$ be a minimal prime of aA such that $\operatorname{ht}(\mathfrak{n}/\mathfrak{q}) = \operatorname{ht}(\mathfrak{n}) - 1$. From the ring inclusion $A/aA \hookrightarrow R/aR$ (because A/aA is injected as a graded summand) and Lemma 5.5, there is a minimal prime $\mathfrak{Q}$ of aR such that $\mathfrak{q} = \mathfrak{Q} \cap A$.

Clearly, $\mathfrak{m} \not\subseteq \mathfrak{Q}$. Indeed, otherwise $(\mathfrak{q}, \mathfrak{m}) \subseteq \mathfrak{Q}$ and since $\mathfrak{m}$ is a prime ideal of R of height at least 1 then $(\mathfrak{q}, \mathfrak{m})$ has height at least 2; this contradicts Krull's Principal Ideal Theorem since $\mathfrak{Q}$ is a minimal prime of a principal ideal.

Let $R' = R/\mathfrak{Q}$ and $A' = A/\mathfrak{q}$. Then R' is a finitely generated graded algebra over A' with $[R']_0 = A'$ and $\mathfrak{m}' := ([R']_+) = \mathfrak{m}R'$. Since $\mathfrak{Q} \not\supseteq \mathfrak{m}$, it follows $\mathrm{ht}(\mathfrak{m}R') \geqslant 1$ and by construction, $\dim(A') = \dim(A) - 1$. So by the inductive hypothesis there is a chain of graded primes $0 = \mathfrak{P}'_0 \subsetneq \cdots \subsetneq \mathfrak{P}'_{k-1}$ in R' such that $\mathfrak{P}'_{k-1} \not\supseteq \mathfrak{m}R'$. Finally, for $j \geqslant 1$ define $\mathfrak{P}_j$ as the inverse image of $\mathfrak{P}'_{j-1}$ via the surjection $R \twoheadrightarrow R'$. □

Recall that $X := \mathrm{Proj}(R)$ is a closed subscheme of $\mathbb{P}^r_A$, for suitable r (= relative embedding dimension of X) whose underlying topological space is the set of all homogeneous prime ideals of R not containing $\mathfrak{m}$ and it has a basis given by the open sets of the form $D_+(f) := \{\wp \in X | f \notin \wp\}$, where $f \in R_+$ is a homogeneous element of positive degree. Here, the sheaf structure is given by the degree zero part of the homogeneous localizations

$$\Gamma\left(D_+(f), \mathcal{O}_X\,|_{D_+(f)}\right) := R_{(f)} = \left\{\frac{g}{f^k} \mid g, f \in R, \deg(g) = k\deg(f)\right\}.$$

Let $K(X) := R_{(0)}$ denote the field of rational functions of X, where

$$R_{(0)} = \left\{\frac{f}{g} \mid f, g \in R, \deg(f) = \deg(g), g \neq 0\right\},$$

the degree zero part of the homogeneous localization of R at the null ideal $(0) \subset R$.

Likewise, denote $\mathbb{P}^s_A = \mathrm{Proj}(S) = \mathrm{Proj}(A[y_0, \ldots, y_s])$.

The dimension $\dim(X)$ of the closed subscheme X is defined to be the supremum of the lenghts of chains of irreducible closed subsets (see, e.g., [66, Definition, p. 5 and p. 86]). The next result is possibly part of the dimensional folklore (cf. [87, Lemma 1.2]).

For any integral domain D, let $\mathrm{Quot}(D)$ denote its field of fractions.

Corollary 5.7. *For the integral subscheme* $X = \mathrm{Proj}(R) \subset \mathbb{P}^r_A$, *we have*

$$\dim(X) = \dim(R) - 1 = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}\left(K(X)\right).$$

Proof. For any prime $\mathfrak{P} \in X$, $\mathfrak{P} + \mathfrak{m}$ is a proper ideal and so $\mathfrak{P}$ is not maximal. Therefore $\mathrm{ht}(\mathfrak{P}) \leqslant \dim(R) - 1$ for any $\mathfrak{P} \in X$, which clearly implies that $\dim(X) \leqslant \dim(R) - 1$.

From Lemma 1.24 we get the equalities

$$\dim(R) = \dim(A) + \mathrm{ht}(\mathfrak{m}) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}(\mathrm{Quot}(R)).$$

There exists a chain of graded prime ideals $0 = \mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_{h-1} \subsetneq \mathfrak{P}_h = \mathfrak{m}$ such that $h = \mathrm{ht}(\mathfrak{m})$ (see, e.g., [110, Theorem 13.7], [19, Theorem 1.5.8]). Let $T = R/\mathfrak{P}_{h-1}$. Since $\mathrm{ht}(\mathfrak{m}T) = 1$, Proposition 5.6 yields the existence of a chain of graded prime ideals $0 = \mathfrak{Q}_0 \subsetneq \cdots \subsetneq \mathfrak{Q}_k$ in T, where $k = \dim(A)$ and $\mathfrak{Q}_k \not\supseteq \mathfrak{m}T$. By taking inverse images along the surjection $R \twoheadrightarrow T$, we obtain a chain of graded prime ideals not containing $\mathfrak{m}$ of length $h - 1 + k = \dim(R) - 1$. Thus, we have the reverse inequality $\dim(X) \geqslant \dim(R) - 1$.

Now, for any $f \in [R]_1$, we have

$$\mathrm{Quot}(R) = R_{(0)}(f)$$

with f transcendental over $K(X) = R_{(0)}$. Therefore

$$\dim(X) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}\left(\mathrm{Quot}(R)\right) - 1 = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}\left(K(X)\right),$$

and so the proof follows. $\square$

Next, we deal with the general multi-graded case, which follows by using an embedding and reducing the problem to the single-graded setting.

Let $T = \bigoplus_{\mathbf{n}\in\mathbb{N}^m} [T]_{\mathbf{n}}$ be a standard m-graded ring over $[T]_{\mathbf{0}} = A$, where $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{N}^m$. The multi-graded irrelevant ideal in this case is given by $\mathfrak{N} = \bigoplus_{n_1>0,\ldots,n_m>0} [T]_{n_1,\ldots,n_m}$. Here, we also assume that T is an integral domain.

Similarly to the single-graded case, we define a multi-projective scheme from T. The multi-projective scheme $\mathrm{MultiProj}(T)$ is given by the set of all multi-homogeneous prime ideals in T which do not contain $\mathfrak{N}$, and its scheme structure is obtained by using multi-homogeneous localizations. The multi-projective scheme $Z := \mathrm{MultiProj}(T)$ is a closed subscheme of $\mathbb{P}_A^{r_1} \times_A \mathbb{P}_A^{r_2} \cdots \times_A \mathbb{P}_A^{r_m}$, for suitable integers $r_1, \ldots, r_m$.

Let $T^{(\Delta)}$ be the single-graded ring $T^{(\Delta)} = \bigoplus_{n\geqslant 0} [T]_{(n,\ldots,n)}$, then the canonical inclusion $T^{(\Delta)} \hookrightarrow T$ induces an isomorphism of schemes $Z = \mathrm{MultiProj}(T) \xrightarrow{\cong} \mathrm{Proj}\left(T^{(\Delta)}\right)$ (see, e.g., [66, Exercise II.5.11]), that corresponds with the Segre embedding

$$\mathbb{P}_A^{r_1} \times_A \mathbb{P}_A^{r_2} \times_A \cdots \times_A \mathbb{P}_A^{r_m} \longrightarrow \mathbb{P}_A^{(r_1+1)(r_2+1)\cdots(r_m+1)-1}.$$

Since we are assuming that Z is an integral scheme, the field of rational functions of Z is given by

$$K(Z) := T_{(0)} = \left\{\frac{f}{g} \mid f, g \in T, \deg(f) = \deg(g), g \neq 0\right\}.$$

The following result yields a multi-graded version of Corollary 5.7.

Corollary 5.8. *For the integral subscheme* $Z = \mathrm{MultiProj}(T) \subset \mathbb{P}_A^{r_1} \times_A \mathbb{P}_A^{r_2} \cdots \times_A \mathbb{P}_A^{r_m}$, *we have*

$$\dim(Z) = \dim(T) - m = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}\left(K(Z)\right).$$

Proof. From the isomorphism $Z \cong \mathrm{Proj}\left(T^{(\Delta)}\right)$ and Corollary 5.7, it follows that

$$\dim(Z) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}\left(K(Z)\right).$$

The dimension formula of Lemma 1.24 gives

$$\dim(T) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}\left(\mathrm{Quot}(T)\right).$$

Choose homogeneous elements $f_1 \in [T]_{(1,0,\ldots,0)}, f_2 \in [T]_{(0,1,\ldots,0)}, \ldots, f_m \in [T]_{(0,0,\ldots,1)}$. Then, we have

$$\mathrm{Quot}(T) = T_{(0)}(f_1, f_2, \ldots, f_m)$$

with $\{f_1, f_2, \ldots, f_m\}$ a transcendence basis over $K(Z) = T_{(0)}$. Therefore

$$\dim(Z) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}(\mathrm{Quot}(T)) - m = \dim(T) - m,$$

and so the result follows. ☐

A generalization for closed subschemes of $\mathbb{P}_A^{r_1} \times_A \mathbb{P}_A^{r_2} \times_A \cdots \times_A \mathbb{P}_A^{r_m}$ is immediate.

Corollary 5.9. *For a closed subscheme* $W = \mathrm{MultiProj}(C) \subset \mathbb{P}_A^{r_1} \times_A \mathbb{P}_A^{r_2} \times_A \cdots \times_A \mathbb{P}_A^{r_m}$*, we have*

$$\dim(W) = \max\left\{ \dim(C/\mathfrak{p}) - m \mid \mathfrak{p} \in W \cap \mathrm{Min}(C) \right\}.$$

Main definitions

We restate the following known concept.

Definition 5.10. *Let* $\mathfrak{R}(X, \mathbb{P}_A^s)$ *denote the set of pairs* (U, φ) *where* U *is an open dense subscheme of* X *and where* $\varphi : U \to \mathbb{P}_A^s$ *is a morphism of* A*-schemes. Two pairs* $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathfrak{R}(X, \mathbb{P}_A^s)$ *are said to be equivalent if there exists an open dense subscheme* $W \subset U_1 \cap U_2$ *such that* $\varphi_1 |_W = \varphi_2 |_W$*. This gives an equivalence relation on* $\mathfrak{R}(X, \mathbb{P}_A^s)$*. A rational map is defined to be an equivalence class in* $\mathfrak{R}(X, \mathbb{P}_A^s)$ *and any element of this equivalence class is said to define the rational map.*

A rational map as above is denoted $\mathcal{F} : X \dashrightarrow \mathbb{P}_A^s$, where the dotted arrow reminds us that typically it will not be defined everywhere as a map. In [65, Lecture 7] (see also [46]) it is explained that, in the case where A is a field the above definition is equivalent to a more usual notion of a rational map in terms of homogeneous coordinate functions. Next, we proceed to show that the same is valid in the relative environment over A.

First it follows from the definition that any morphism $U \to \mathbb{P}_A^s$ as above from an open dense subset defines a unique rational map $X \dashrightarrow \mathbb{P}_A^s$. Now, let there be given $s+1$ forms $\mathbf{f} = \{f_0, f_1, \ldots, f_s\} \subset R$ of the same degree $d > 0$. Let $\mathfrak{h} : S \to R$ be the graded homomorphism of A-algebras given by

$$\begin{aligned} \mathfrak{h} : S = A[y_0, y_1, \ldots, y_s] &\longrightarrow R \\ y_i &\mapsto f_i. \end{aligned}$$

There corresponds to it a morphism of A-schemes

$$\Phi(\mathbf{f}) = \mathrm{Proj}(\mathfrak{h}) : D_+(\mathbf{f}) \longrightarrow \mathrm{Proj}(S) = \mathbb{P}_A^s$$

where $D_+(\mathbf{f}) \subset \mathrm{Proj}(R) = X$ is the open subscheme given by

$$D_+(\mathbf{f}) = \bigcup_{i=0}^{s} D_+(f_i).$$

Therefore, a set of $s+1$ forms $\mathbf{f} = \{f_0, f_1, \ldots, f_s\} \subset R$ of the same positive degree determines a unique rational map given by the equivalence class of $(D_+(\mathbf{f}), \Phi(\mathbf{f}))$ in $\mathfrak{R}(X, \mathbb{P}^s_A)$.

Definition 5.11. *We call $\Phi(\mathbf{f})$ the $\mathbf{f}$-coordinate morphism and denote the corresponding rational map by $\mathcal{F}_{\mathbf{f}}$.*

Conversely:

Lemma 5.12. *Any rational map $\mathcal{F} : X = \mathrm{Proj}(R) \dashrightarrow \mathbb{P}^s_A$ is of the form $\mathcal{F}_{\mathbf{f}}$, where $\mathbf{f}$ are forms of the same positive degree.*

Proof. Let U be an open dense subset in X and $\varphi : U \to \mathbb{P}^s_A$ be a morphism, such that the equivalence class of the pair (U, φ) in $\mathfrak{R}(X, \mathbb{P}^s_A)$ is equal to $\mathcal{F}$.

Consider $V = D_+(y_0)$ and $W = \varphi^{-1}(V)$ and restrict to an affine open subset, $W' = \mathrm{Spec}(R_{(\ell)}) \subset W$, where $\ell \in R$ is a homogeneous element of positive degree. It yields a morphism $\varphi\,|_{W'} : W' \to V$, that corresponds to a ring homomorphism

$$\tau : S_{(y_0)} \to R_{(\ell)},$$

where $T_{(h)}$ stands for the degree zero part of the homogeneous localization of a graded ring T at the powers of a homogeneous element $h \in T$.

For each $0 < i \leqslant s$ we have

$$\tau\left(\frac{y_i}{y_0}\right) = \frac{g_i}{\ell^{\alpha_i}}$$

where $\deg(g_i) = \alpha_i \deg(\ell)$. Setting $\alpha := \max_{1 \leqslant i \leqslant s}\{\alpha_i\}$, we write

$$f_0 := \ell^{\alpha} \quad \text{and} \quad f_i := \ell^{\alpha}\frac{g_i}{\ell^{\alpha_i}} = \ell^{\alpha-\alpha_i} g_i \text{ for } 1 \leqslant i \leqslant s.$$

By construction, $\varphi\,|_{W'} = \Phi(\mathbf{f})\,|_{W'}$, where $\Phi(\mathbf{f})$ denotes the $\mathbf{f}$-coordinate morphism determined by $\mathbf{f} = \{f_0, \ldots, f_s\}$, as in definition Definition 5.11, hence $\mathcal{F} = \mathcal{F}_{\mathbf{f}}$ where $\mathbf{f} = \{f_0, \ldots, f_s\}$. □

Given a rational map $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$, any ordered $(s+1)$-tuple $\mathbf{f} = \{f_0, f_1, \ldots, f_s\}$ of forms of the same positive degree such that $\mathcal{F} = \mathcal{F}_{\mathbf{f}}$ is called a *representative* of the rational map $\mathcal{F}$.

The following result explains the flexibility of representatives of the same rational map.

Lemma 5.13. *Let $\mathbf{f} = \{f_0, \ldots, f_s\}$ and $\mathbf{f}' = \{f'_0, \ldots, f'_s\}$ stand for representatives of a rational map $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$. Then $(f_0 : \cdots : f_s)$ and $(f'_0 : \cdots : f'_s)$ are proportional coordinate sets in the sense that there exist homogeneous forms h, h' of positive degree such that $hf'_i = h'f_i$ for $i = 0, \ldots, s$.*

Proof. Proceed similarly to Lemma 5.12. Let $\Phi(\mathbf{f}) : D_+(\mathbf{f}) \to \mathbb{P}^s_A$ and $\Phi(\mathbf{f}') : D_+(\mathbf{f}') \to \mathbb{P}^s_A$ be morphisms as in Definition 5.11. Let $V = \operatorname{Spec}(D_+(y_0))$ and choose $W = \operatorname{Spec}(R_{(\ell)})$ such that $W \subset \Phi(\mathbf{f})^{-1}(V) \cap \Phi(\mathbf{f}')^{-1}(V)$ and $\Phi(\mathbf{f})|_W = \Phi(\mathbf{f}')|_W$.

The morphisms $\Phi(\mathbf{f})|_W : W \to V$ and $\Phi(\mathbf{f}')|_W : W \to V$ correspond with the ring homomorphisms $\tau : S_{(y_0)} \to R_{(\ell)}$ and $\tau' : S_{(y_0)} \to R_{(\ell)}$ such that

$$\tau\left(\frac{y_i}{y_0}\right) = \frac{f_i}{f_0} \quad \text{and} \quad \tau'\left(\frac{y_i}{y_0}\right) = \frac{f'_i}{f'_0},$$

respectively. Since this is now an affine setting, the ring homomorphisms τ and τ' are the same (see e.g. [57, Theorem 2.35], [66, Proposition II.2.3]). It follows that, for every $i = 0, \ldots, s$, $f'_i/f'_0 = f_i/f_0$ as elements of the field of fractions of R. Therefore, there are homogeneous elements $h, h' \in R$ ($h = f_0, h' = f'_0$) such that $hf'_i = h'f_i$ for $i = 0, \ldots, s$. The claim now follows. □

In the above notation, we often denote $\mathcal{F}_{\mathbf{f}}$ simply by $(f_0 : \cdots : f_s)$ and use this symbol for a representative of $\mathcal{F}$.

Remark 5.14. *Note that the identity morphism of $\mathbb{P}^r_A$ is a rational map of $\mathbb{P}^r_A$ to itself with natural representative $(x_0 : \cdots : x_r)$ where $\mathbb{P}^r_A = \operatorname{Proj}(A[x_0, \ldots, x_r])$. Similarly, the identity morphism of $X = \operatorname{Proj}(R)$ is a rational map represented by $(x_0 : \cdots : x_r)$, where now $x_0, \ldots, x_r$ generate the A-module $[R]_1$, and it is denoted by Id_X.*

The following sums up a version of [129, Proposition 1.1] over an integral domain. Due to Lemma 5.13, the proof is a literal transcription of the proof in loc. cit.

Proposition 5.15. *Let $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$ be a rational map with representative $\mathbf{f}$. Set $I = (\mathbf{f})$. Then, the following statements hold:*

(i) The representatives of $\mathcal{F}$ correspond bijectively to the non-zero homogeneous vectors in the rank one graded R-module $\operatorname{Hom}_R(I, R)$.

(ii) If $\operatorname{grade}(I) \geqslant 2$, any representative of $\mathcal{F}$ is a multiple of $\mathbf{f}$ by a homogeneous element in R.

Proof. (i) Follows from Lemma 5.13 and the isomorphism $\operatorname{Hom}_R(I, R) \cong \left(R :_{\operatorname{Quot}(R)} I\right)$.

(ii) The condition $\operatorname{grade}(I) \geqslant 2$ is equivalent to $\operatorname{Hom}_R(I, R) \cong R$ (see e.g. [19, Exercise 1.2.24]). □

Remark 5.16. *If R is in addition an UFD then any rational map has a unique representative up to a multiplier – this is the case, e.g., when A is a UFD and R is a polynomial ring over A.*

One more notational convention: if $\mathbf{f} = \{f_0, \ldots, f_s\}$ are forms of the same degree, $A[\mathbf{f}]$ will denote the A-subalgebra of R generated by these forms.

An important immediate consequence is as follows:

Corollary 5.17. *Let* $\mathbf{f} = (f_0 : \cdots : f_s)$ *and* $\mathbf{f}' = (f'_0 : \cdots : f'_s)$ *stand for representatives of the same rational map* $\mathcal{F} : X = \mathrm{Proj}(R) \dashrightarrow \mathbb{P}^s_A$*. Then* $A[\mathbf{f}] \cong A[\mathbf{f}']$ *as graded* A*-algebras and* $\mathcal{R}_R(I) \cong \mathcal{R}_R(I')$ *as bigraded* $\mathfrak{A}$*-algebras, where* $I = (\mathbf{f})$ *and* $I' = (\mathbf{f}')$*.*

Proof. Let $\mathcal{J}$ and $\mathcal{J}'$ respectively denote the ideals of defining equations of $\mathcal{R}_R(I)$ and $\mathcal{R}_R(I')$, as given in (5.1). From Lemma 5.13, there exist homogeneous elements $h, h' \in R$ such that $hf'_i = h'f_i$ for $i = 0, \ldots, s$. Clearly, then $I \cong I'$ have the same syzygies, hence the defining ideals $\mathcal{L}$ and $\mathcal{L}'$ of the respective symmetric algebras coincide. Since R is an integral domain and I and I' are nonzero, then Lemma 1.10 yields

$$\mathcal{J} = \mathcal{L} : I^\infty = \mathcal{L}' : I'^\infty = \mathcal{J}'.$$

Therefore, $\mathcal{R}_R(I) \cong \mathfrak{A}/\mathcal{J} = \mathfrak{A}/\mathcal{J}' \cong \mathcal{R}_R(I')$ as bigraded $\mathfrak{A}$-algebras. Consequently,

$$A[\mathbf{f}] \cong \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I) \cong \mathcal{R}_R(I')/\mathfrak{m}\mathcal{R}_R(I') \cong A[\mathbf{f}']$$

as graded A-algebras. $\square$

Image, degree and birational maps

This part is essentially a recap on the algebraic description of the image, the degree and the birationality of a rational map in the relative case. Most of the material here has been considered in a way or another as a previsible extension of the base field situation (see, e.g., [23, Theorem 2.1]).

Definition-Proposition 5.18. *Let* $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$ *be a rational map. The image of* $\mathcal{F}$ *is equivalently defined as:*

(I1) *The closure of the image of a morphism* $U \to \mathbb{P}^s_A$ *defining* $\mathcal{F}$*, for some (any) open dense subset* $U \subset X$*.*

(I2) *The closure of the image of the* $\mathbf{f}$*-coordinate morphism* $\Phi(\mathbf{f})$*, for some (any) representative* $\mathbf{f}$ *of* $\mathcal{F}$*.*

(I3) $\mathrm{Proj}\,(A[\mathbf{f}])$*, for some (any) representative* $\mathbf{f}$ *of* $\mathcal{F}$*, up to degree normalization of* $A[\mathbf{f}]$*.*

Proof. The equivalence of (I1) and (I2) is clear by the previous developments. To check that (I2) and (I3) are equivalent, consider the ideal sheaf $\mathcal{J}$ given as the kernel of the canonical homomorphism

$$\mathcal{O}_{\mathbb{P}^s_A} \to \Phi(\mathbf{f})_* \mathcal{O}_{D_+(\mathbf{f})}$$

It defines a closed subscheme $Y \subset \mathbb{P}^s_A$ which corresponds with the schematic image of $\Phi(\mathbf{f})$ (see, e.g., [57, Proposition 10.30]). The underlying topological space of Y coincides with the closure of the image of $\Phi(\mathbf{f})$. Now, for any $0 \leqslant i \leqslant s$, $\mathcal{O}_{\mathbb{P}^s_A}(D_+(y_i)) = S_{(y_i)}$ and $\left(\Phi(\mathbf{f})_* \mathcal{O}_{D_+(\mathbf{f})}\right)(D_+(y_i)) = R_{(f_i)}$. Then, for $0 \leqslant i \leqslant s$, there is an exact sequence

$$0 \to \mathcal{J}\,(D_+(y_i)) \to S_{(y_i)} \to R_{(f_i)}.$$

Thus, $\mathcal{J}(D_+(y_i)) = J_{(y_i)}$ for any $0 \leqslant i \leqslant s$, where J is the kernel of the A-algebra homomorphism $\alpha : S \to A[\mathbf{f}] \subset R$ given by $y_i \mapsto f_i$. This implies that $\mathcal{J}$ is the sheafification of J. Therefore, $Y \cong \mathrm{Proj}(S/J) \cong \mathrm{Proj}(A[\mathbf{f}])$. □

Now we consider the degree of a rational map $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$. By Definition-Proposition 5.18, the field of rational functions of the image Y of $\mathcal{F}$ is

$$K(Y) = A[\mathbf{f}]_{(0)},$$

where $\mathbf{f} = (f_0 : \cdots : f_s)$ is a representative of $\mathcal{F}$. Here, $A[\mathbf{f}]$ is naturally A-graded as an A-subalgebra of R, but we may also consider it as a standard A-graded algebra by a degree normalization.

We get a natural field extension $K(Y) = A[\mathbf{f}]_{(0)} \hookrightarrow R_{(0)} = K(X)$.

Definition 5.19. *The degree of* $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$ *is*

$$\deg(\mathcal{F}) := [K(X) : K(Y)] .$$

We say that $\mathcal{F}$ is generically finite if $[K(X) : K(Y)] < \infty$. If the field extension $K(X)|K(Y)$ is infinite, we agree to say that $\mathcal{F}$ has no well-defined degree (also, in this case, we often say that $\deg(\mathcal{F}) = 0$).

The following properties are well-known over a coefficient field. Its restatement in the relative case is for the reader's convenience.

Proposition 5.20. *Let* $\mathcal{F} : X \dashrightarrow \mathbb{P}^s_A$ *be a rational map with image* $Y \subset \mathbb{P}^s_A$.

(i) Let $\mathbf{f}$ *denote a representative of* $\mathcal{F}$ *and let* $\Phi(\mathbf{f})$ *be the associated* $\mathbf{f}$*-coordinate morphism. Then,* $\mathcal{F}$ *is generically finite if and only if there exists an open dense subset* $U \subset Y$ *such that* $\Phi(\mathbf{f})^{-1}(U) \to U$ *is a finite morphism.*

(ii) $\mathcal{F}$ *is generically finite if and only if* $\dim(X) = \dim(Y)$.

Proof. (i) Let $\Phi(\mathbf{f}) : D_+(\mathbf{f}) \subset X \to Y \subset \mathbb{P}^s_A$ be the $\mathbf{f}$-coordinate morphism of $\mathcal{F}$. One has an equality of fields of rational functions $K(X) = K(D_+(\mathbf{f}))$. But on $D_+(\mathbf{f})$ the rational map $\mathcal{F}$ is defined by a morphism, in which case the result is given in [66, Exercise II.3.7].

(ii) By Corollary 5.7 we have $\dim(X) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}(K(X))$ and by the same token, $\dim(Y) = \dim(A) + \mathrm{trdeg}_{\mathrm{Quot}(A)}(K(Y))$. It follows that

$$\dim(X) = \dim(Y) \iff \mathrm{trdeg}_{\mathrm{Quot}(A)}(K(X)) = \mathrm{trdeg}_{\mathrm{Quot}(A)}(K(Y)) .$$

The later condition is equivalent to $\mathrm{trdeg}_{K(Y)}(K(X)) = 0$, and so the proof follows. □

Next we define birational maps in the relative environment over A. While any of the three alternatives below sounds equally fit as a candidate (as a deja vu of the classical coefficient field setup), showing that they are in fact mutually equivalent requires a small bit of work.

Definition-Proposition 5.21. *Let* $\mathcal{F} : X \subset \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A$ *be a rational map with image* $Y \subset \mathbb{P}^s_A$*. The map* $\mathcal{F}$ *is said to be birational onto its image if one of the following equivalent conditions is satisfied:*

(B1) $\deg(\mathcal{F}) = 1$*, that is* $K(X) = K(Y)$.

(B2) *There exists some open dense subset* $U \subset X$ *and a morphism* $\varphi : U \to \mathbb{P}^s_A$ *such that the pair* (U, φ) *defines* $\mathcal{F}$ *and such that* φ *is an isomorphism onto an open dense subset* $V \subset Y$.

(B3) *There exists a rational map* $\mathcal{G} : Y \subset \mathbb{P}^s_A \dashrightarrow X \subset \mathbb{P}^r_A$ *such that, for some (any) representative* $\mathbf{f}$ *of* $\mathcal{F}$ *and some (any) representative* $\mathbf{g} = (g_0 : \cdots : g_r)$ *of* $\mathcal{G}$*, the composite*

$$\mathbf{g}(\mathbf{f}) = (g_0(\mathbf{f}) : \cdots : g_r(\mathbf{f}))$$

is a representative of the identity rational map on X.

Proof. (B1) $\Rightarrow$ (B2). Let $\varphi' : U' \to \mathbb{P}^s_A$ be a morphism from an open dense subset $U' \subset X$ such that (U', φ') defines $\mathcal{F}$. Let η denote the generic point of X and ξ that of Y. The field inclusion $\mathcal{O}_{Y,\xi} \cong K(Y) \hookrightarrow K(X) \cong \mathcal{O}_{X,\eta}$ coincides with the induced local ring homomorphism

$$(\varphi')^\sharp_\eta : \mathcal{O}_{Y,\xi} \to \mathcal{O}_{X,\eta}.$$

Since by assumption $\deg(\mathcal{F}) = 1$, $(\varphi')^\sharp_\eta$ is an isomorphism. Then, by [57, Proposition 10.52] $(\varphi')^\sharp_\eta$ "extends" to an isomorphism from an open neighborhood U of η in X onto an open neighborhood V of ξ in Y. Now, take the restriction $\varphi = \varphi' |_U : U \xrightarrow{\cong} V$ as the required isomorphism.

(B2) $\Rightarrow$ (B3) Let $\varphi : U \subset X \xrightarrow{\cong} V \subset Y$ be a morphism defining $\mathcal{F}$, which is an isomorphism from an open dense subset $U \subset X$ onto an open dense subset $V \subset Y$. Let $\psi = \varphi^{-1} : V \subset Y \xrightarrow{\cong} U \subset X$ be the inverse of φ. Let $\mathcal{G} : Y \subset \mathbb{P}^s_A \dashrightarrow X \subset \mathbb{P}^r_A$ be the rational map defined by (V, ψ).

Let Id_X be the identity rational map on X (Remark 5.14). Take any representatives $\mathbf{f} = (f_0 : \cdots : f_s)$ of $\mathcal{F}$ and $\mathbf{g} = (g_0 : \cdots : g_r)$ of $\mathcal{G}$. Let $\mathcal{G} \circ \mathcal{F}$ be the composition of $\mathcal{F}$ and $\mathcal{G}$, i.e. the rational map defined by $(U, \psi \circ \varphi)$. Since $\psi \circ \varphi$ is the identity morphism on U, Definition 5.10 implies that the pair $(U, \psi \circ \varphi)$ gives the equivalence class of Id_X. Thus, we have $\mathrm{Id}_X = \mathcal{G} \circ \mathcal{F}$, and by construction $\mathbf{g}(\mathbf{f})$ is a representative of $\mathcal{G} \circ \mathcal{F}$.

(B3) $\Rightarrow$ (B1) Take a representative $(f_0 : \cdots : f_s)$ of $\mathcal{F}$ and let $\mathcal{G}$ and $(g_0 : \cdots : g_r)$ be as in the assumption. Since the identity map of X is defined by the representative $(x_0 : \cdots : x_r)$, where $[R]_1 = Ax_0 + \cdots + Ax_r$ (see Remark 5.14), then Lemma 5.13 yields the existence of nonzero (homogeneous) $h, h' \in R$ such that $h \cdot g_i(\mathbf{f}) = h' \cdot x_i$, for $i = 0, \ldots, r$. Then, for suitable $e \geqslant 0$,

$$\frac{x_i}{x_0} = \frac{g_i(\mathbf{f})}{g_0(\mathbf{f})} = \frac{f_0^e\,(g_i(f_1/f_0, \ldots, f_m/f_0))}{f_0^e\,(g_0(f_1/f_0, \ldots, f_m/f_0))} = \frac{g_i(f_1/f_0, \ldots, f_m/f_0)}{g_0(f_1/f_0, \ldots, f_m/f_0)}, \quad i = 0, \ldots, r$$

This shows the reverse inclusion $K(X) \subset K(Y)$. $\square$

The graph of a rational map

The tensor product $\mathfrak{A} := R \otimes_A A[\mathbf{y}] \cong R[\mathbf{y}]$ has a natural structure of a standard bigraded A-algebra. Accordingly, the fiber product $\mathrm{Proj}(R) \times_A \mathbb{P}^s_A$ has a natural structure of a biprojective scheme over $\mathrm{Spec}(A)$. Thus, $\mathrm{Proj}(R) \times_A \mathbb{P}^s_A = \mathrm{BiProj}(\mathfrak{A})$.

The graph of a rational map $\mathcal{F} : X = \mathrm{Proj}(R) \dashrightarrow \mathbb{P}^s_A$ is a subscheme of this structure, in the following way:

Definition-Proposition 5.22. *The graph of $\mathcal{F}$ is equivalently defined as:*

(G1) *The closure of the image of the morphism $(\iota, \varphi) : U \to X \times_A \mathbb{P}^s_A$, where $\iota : U \hookrightarrow X$ is the natural inclusion and $\varphi : U \to \mathbb{P}^s_A$ is a morphism from some (any) open dense subset defining $\mathcal{F}$.*

(G2) *For some (any) representative $\mathbf{f}$ of $\mathcal{F}$, the closure of the image of the morphism $(\iota, \Phi(\mathbf{f})) : D_+(\mathbf{f}) \longrightarrow X \times_A \mathbb{P}^s_A$, where $\iota : D_+(\mathbf{f}) \hookrightarrow X$ is the natural inclusion and $\Phi(\mathbf{f}) : D_+(\mathbf{f}) \to \mathbb{P}^s_A$ is the $\mathbf{f}$-coordinate morphism.*

(G3) $\mathrm{BiProj}\,(\mathcal{R}_R(I))$*, where $I = (\mathbf{f})$ for some (any) representative $\mathbf{f}$ of $\mathcal{F}$.*

Proof. The equivalence of (G1) and (G2) is clear, so we proceed to show that (G2) and (G3) give the same scheme. Recall that, as in (5.1), the Rees algebra of an ideal such as I is a bigraded $\mathfrak{A}$-algebra. The proof follows the same steps of the argument for the equivalence of (I2) and (I3) in the definition of the image of $\mathcal{F}$ (cf. Definition-Proposition 5.18).

Let $\Gamma(\mathbf{f})$ denote the morphism as in (G2) and let $\mathfrak{G} \subset X \times_A \mathbb{P}^s_A$ denote its schematic image. The underlying topological space of $\mathfrak{G}$ coincides with the closure of the image of $\Gamma(\mathbf{f})$. Then, the ideal sheaf of $\mathfrak{G}$ is the kernel $\mathfrak{J}$ of the corresponding homomorphism of ring sheaves

$$\mathcal{O}_{X \times_A \mathbb{P}^s_A} \to \Gamma(\mathbf{f})_* \mathcal{O}_{D_+(\mathbf{f})}. \tag{5.2}$$

Since the irrelevant ideal of $\mathfrak{A}$ is $([R]_1) \cap (\mathbf{y})$, by letting $[R]_1 = Ax_0 + \cdots + Ax_r$ we can see that an affine open cover is given by $\mathrm{Spec}\left(\mathfrak{A}_{(x_i y_j)}\right)$ for $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$, where $\mathfrak{A}_{(x_i y_j)}$ denotes the degree zero part of the bihomogeneous localization at powers of $x_i y_j$, to wit

$$\mathfrak{A}_{(x_i y_j)} = \left\{ \frac{g}{(x_i y_j)^\alpha} \mid g \in \mathfrak{A} \text{ and } \mathrm{bideg}(g) = (\alpha, \alpha) \right\}. \tag{5.3}$$

We have $\mathcal{O}_{X \times_A \mathbb{P}^s_A}\left(D_+(x_i y_j)\right) = \mathfrak{A}_{(x_i y_j)}$ and $\left(\Gamma(\mathbf{f})_* \mathcal{O}_{D_+(\mathbf{f})}\right)\left(D_+(x_i y_j)\right) = R_{(x_i f_j)}$, for $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$. Then (5.2) yields the exact sequence

$$0 \to \mathfrak{J}\left(D_+(x_i y_j)\right) \to \mathfrak{A}_{(x_i y_j)} \to R_{(x_i f_j)}.$$

Let $\mathcal{J}$ be the kernel of the homomorphism of bigraded $\mathfrak{A}$-algebras $\mathfrak{A} \to \mathcal{R}_R(I) \subset R[t]$ given by $y_i \mapsto f_i t$. The fact that $\mathcal{R}_R(I)_{(x_i f_j t)} \cong R_{(x_i f_j)}$, yields the equality $\mathfrak{J}\left(D_+(x_i y_j)\right) = \mathcal{J}_{(x_i y_j)}$. It follows that $\mathfrak{J}$ is the sheafification of $\mathcal{J}$. Therefore, $\mathfrak{G} \cong \mathrm{BiProj}(\mathfrak{A}/\mathcal{J}) \cong \mathrm{BiProj}(\mathcal{R}_R(I))$. $\square$

Saturated fiber cones over an integral domain

In this part we introduce the notion of a saturated fiber cone over an integral domain, by closely lifting from the ideas in Chapter 3. As will be seen, the notion is strongly related to the degree and birationality of rational maps.

For simplicity, assume that $R = A[\mathbf{x}] = A[x_0, \ldots, x_r]$, a standard graded polynomial ring over A and set $\mathbb{K} := \mathrm{Quot}(A), \mathfrak{m} = (x_0, \ldots, x_r)$.

The central object is the following graded A-algebra

$$\widetilde{\mathfrak{F}_R(I)} := \bigoplus_{n=0}^{\infty} \left[\left(I^n : \mathfrak{m}^\infty\right)\right]_{nd},$$

which we call the *saturated fiber cone* of I (Definition 3.3).

Note the natural inclusion of graded A-algebras $\mathfrak{F}_R(I) \subset \widetilde{\mathfrak{F}_R(I)}$.

For any $i \geqslant 0$, the local cohomology module $H_\mathfrak{m}^i(\mathcal{R}_R(I))$ has a natural structure of bigraded $\mathcal{R}_R(I)$-module, which comes out of the fact that $H_\mathfrak{m}^i(\mathcal{R}_R(I)) = H_{\mathfrak{m}\mathcal{R}_R(I)}^i(\mathcal{R}_R(I))$ (see also [29, Lemma 2.1]). In particular, each R-graded part

$$\left[H_\mathfrak{m}^i(\mathcal{R}_R(I))\right]_j$$

has a natural structure of graded $\mathfrak{F}_R(I)$-module.

Let $\mathrm{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))$ denote the Rees algebra $\mathcal{R}_R(I)$ viewed as a "one-sided" graded R-algebra.

Lemma 5.23. *With the above notation, we have:*

(i) There is an isomorphism of graded A-algebras

$$\widetilde{\mathfrak{F}_R(I)} \;\cong\; H^0\left(\mathrm{Proj}_{R\text{-gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\mathrm{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))}\right).$$

(ii) $\widetilde{\mathfrak{F}_R(I)}$ is a finitely generated graded $\mathfrak{F}_R(I)$-module.

(iii) There is an exact sequence

$$0 \rightarrow \mathfrak{F}_R(I) \rightarrow \widetilde{\mathfrak{F}_R(I)} \rightarrow \left[H_\mathfrak{m}^1(\mathcal{R}_R(I))\right]_0 \rightarrow 0$$

of finitely generated graded $\mathfrak{F}_R(I)$-modules.

(iv) If $A \rightarrow A'$ is a flat ring homomorphism, then there is an isomorphism of graded A'-algebras

$$\widetilde{\mathfrak{F}_R(I)} \otimes_A A' \cong \widetilde{\mathfrak{F}_{R'}(IR')},$$

where $R' = R \otimes_A A'$.

Proof. (i) Since $\mathcal{R}_R(I) \cong \bigoplus_{n=0}^{\infty} I^n(nd)$, by computing Čech cohomology with respect to the affine open covering $\left(\text{Spec}\left(\mathcal{R}_R(I)_{(x_i)}\right)\right)_{0\leqslant i\leqslant r}$ of $\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))$, we obtain

$$\begin{aligned} H^0\left(\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))}\right) &\cong \bigoplus_{n\geqslant 0} H^0\left(\text{Proj}(R), (I^n)^{\sim}(nd)\right) \\ &= \bigoplus_{n=0}^{\infty} \left[\left(I^n : \mathfrak{m}^{\infty}\right)\right]_{nd} \quad \text{([66, Exercise II.5.10]).} \\ &= \widetilde{\mathfrak{F}_R(I)}. \end{aligned}$$

(ii) and (iii) From [87, Corollary 1.5] (see also [47, Theorem A4.1]) and the fact that $H^0_{\mathfrak{m}}(\mathcal{R}_R(I)) = 0$, there is an exact sequence

$$0 \to [\mathcal{R}_R(I)]_0 \to H^0(\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))}) \cong \widetilde{\mathfrak{F}_R(I)} \to \left[H^1_{\mathfrak{m}}(\mathcal{R}_R(I))\right]_0 \to 0$$

of $\mathfrak{F}_R(I)$-modules. Now $\left[H^1_{\mathfrak{m}}(\mathcal{R}_R(I))\right]_0$ is a finitely generated module over $\mathfrak{F}_R(I)$ (see, e.g., Proposition 3.7, [26, Theorem 2.1]), thus implying that $\widetilde{\mathfrak{F}_R(I)}$ is also finitely generated over $\mathfrak{F}_R(I)$.

(iv) Since $A \to A'$ is flat, a base change yields

$$H^0\left(B, \mathcal{O}_B\right) \cong H^0\left(\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))}\right) \otimes_A A',$$

where $B = \text{Proj}_{R\text{-gr}}(\mathcal{R}_R(I)) \times_A A' = \text{Proj}_{R\text{-gr}}\left(\mathcal{R}_R(I) \otimes_A A'\right)$. Also $\mathcal{R}_R(I) \otimes_A A' \cong \mathcal{R}_{R'}(IR')$, by flatness, hence the result follows. $\square$

Let $\mathcal{F} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A$ be a rational map with representative $\mathbf{f} = (f_0 : \cdots : f_s)$. Let $\mathcal{G} : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow \mathbb{P}^s_{\mathbb{K}}$ denote a rational map with representative $\mathbf{f}$, where each f_i is considered as an element of $\mathbb{K}[\mathbf{x}]$. Set $I = (\mathbf{f}) \subset R$.

Remark 5.24. *The rational map $\mathcal{F}$ is generically finite if and only if the rational map $\mathcal{G}$ is so, and we have the equality* $\deg(\mathcal{F}) = \deg\left(\mathcal{G}\right)$ *. In fact, let Y and Z be the images of $\mathcal{F}$ and $\mathcal{G}$, respectively. Since* $K(\mathbb{P}^r_A) = R_{(0)} = \mathbb{K}[\mathbf{x}]_{(0)} = K(\mathbb{P}^r_{\mathbb{K}})$ *and* $K(Y) = A[\mathbf{f}]_{(0)} = \mathbb{K}[\mathbf{f}]_{(0)} = K(Z)$*, then the statement follows from Definition 5.19 and Proposition 5.20.*

The following result is a simple consequence of Theorem 3.4.

Theorem 5.25. *Suppose that $\mathcal{F}$ is generically finite. Then, the following statements hold:*

(i) $\deg(\mathcal{F}) = \left[\widetilde{\mathfrak{F}_R(I)} : \mathfrak{F}_R(I)\right]$.

(ii) $e\left(\widetilde{\mathfrak{F}_R(I)} \otimes_A \mathbb{K}\right) = \deg(\mathcal{F}) \cdot e\left(\mathfrak{F}_R(I) \otimes_A \mathbb{K}\right)$*, where $e(-)$ stands for multiplicity.*

(iii) Under the additional condition of $\mathfrak{F}_R(I)$ *being integrally closed, then* $\mathcal{F}$ *is birational if and only if* $\widetilde{\mathfrak{F}_R(I)} = \mathfrak{F}_R(I)$.

Proof. (i) Let $\mathcal{G} : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow \mathbb{P}^s_{\mathbb{K}}$ be the rational map as above. Since $A \hookrightarrow \mathbb{K}$ is flat, $\mathfrak{F}_R(I) \otimes_A \mathbb{K} \cong \mathfrak{F}_{\mathbb{K}[\mathbf{x}]}(I\mathbb{K}[\mathbf{x}])$ and $\widetilde{\mathfrak{F}_R(I)} \otimes_A \mathbb{K} \cong \widetilde{\mathfrak{F}_{\mathbb{K}[\mathbf{x}]}(I\mathbb{K}[\mathbf{x}])}$ (Lemma 5.23 (iv)).

Thus from Theorem 3.4, we obtain $\deg(\mathcal{G}) = \left[\widetilde{\mathfrak{F}_R(I)} \otimes_A \mathbb{K} : \mathfrak{F}_R(I) \otimes_A \mathbb{K}\right]$. It is clear that $\left[\widetilde{\mathfrak{F}_R(I)} \otimes_A \mathbb{K} : \mathfrak{F}_R(I) \otimes_A \mathbb{K}\right] = \left[\widetilde{\mathfrak{F}_R(I)} : \mathfrak{F}_R(I)\right]$. Finally, Remark 5.24 yields the equality $\deg(\mathcal{F}) = \deg(\mathcal{G})$.

(ii) It follows from the associative formula for multiplicity (see, e.g., [19, Corollary 4.7.9]).

(iii) It suffices to show that, assuming that $\mathcal{F}$ is birational onto the image and that $\mathfrak{F}_R(I)$ is integrally closed, then $\widetilde{\mathfrak{F}_R(I)} = \mathfrak{F}_R(I)$. Since $\deg(\mathcal{F}) = 1$, part (i) gives

$$\mathrm{Quot}\left(\widetilde{\mathfrak{F}_R(I)}\right) = \mathrm{Quot}\left(\mathfrak{F}_R(I)\right).$$

Since $\mathfrak{F}_R(I)$ is integrally closed and $\mathfrak{F}_R(I) \hookrightarrow \widetilde{\mathfrak{F}_R(I)}$ is an integral extension (see Lemma 5.23(ii)), then $\widetilde{\mathfrak{F}_R(I)} = \mathfrak{F}_R(I)$. □

5.3 Additional algebraic tools

In this section we gather a few algebraic tools to be used in the specialization of rational maps. The section is divided in two subsections, and each subsection deals with a different theme that is important on its own.

Grade of certain generic determinantal ideals

We provide lower bounds for the grade of certain generic determinantal ideals. This generic situation will later be specialized in Section 5.5.

In this subsection we agree to change the previous notation, by letting R denote an arbitrary Noetherian ring.

The next lemma deals with generic ideals deforming ideals in R (see, e.g., [132, Proposition 3.2] for a similar setup).

Lemma 5.26. *Let* $\mathbf{z} = (z_{i,j})$ *be a new set of variables with* $1 \leqslant i \leqslant n$ *and* $1 \leqslant j \leqslant m$ *and* S *be the polynomial ring* $S = R[\mathbf{z}]$. *Let* $I = (f_1, \ldots, f_m) \subset R$ *be an ideal. Let* J *be the ideal* $J = (p_1, p_2, \ldots, p_n) \subset R[\mathbf{z}]$ *such that*

$$p_i = f_1 z_{i,1} + f_2 z_{i,2} + \cdots + f_m z_{i,m}.$$

Then $\mathrm{grade}(J) \geqslant \min\{n, \mathrm{grade}(I)\}$.

Proof. Let Q be a prime ideal containing J. If Q contains all the f_i's, then $\operatorname{depth}(S_Q) \geqslant \operatorname{grade}(I)$. Otherwise, say, $f_1 \notin Q$. Then, we can write

$$\frac{p_i}{f_1} = z_{i,1} + \frac{f_2}{f_1} z_{i,2} + \cdots + \frac{f_m}{f_1} z_{i,m} \in R_{f_1}[\mathbf{z}]$$

as elements of the localization $S_{f_1} = R_{f_1}[\mathbf{z}]$. Since $\{z_{1,1}, \ldots, z_{n,1}\}$ is a regular sequence in $R_{f_1}[\mathbf{z}]$, then so is the sequence $\{p_1/f_1, \ldots, p_n/f_1\}$. Then, clearly $\{p_1, \ldots, p_n\}$ is a regular sequence in $R_{f_1}[\mathbf{z}]$, hence $\operatorname{depth}(S_Q) \geqslant \operatorname{grade}(JR_{f_1}[\mathbf{z}]) \geqslant n$. □

The next proposition is now an easy routine procedure of inverting-localizing at a suitable entry. We give the proof for the sake of completeness.

Proposition 5.27. *Let* $I_j = (f_{j,1}, \ldots, f_{j,m_j}) \subset R$ *be ideals for* $1 \leqslant j \leqslant s$. *Set* $g = \min_{1 \leqslant j \leqslant s}\{\operatorname{grade}(I_j)\}$. *Let* $\mathbf{z} = (z_{i,j,k})$ *be a new set of variables with* $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$ *and* $1 \leqslant k \leqslant m_j$. *Let* S *be the polynomial ring* $S = R[\mathbf{z}]$. *Let* M *be the* $r \times s$ *matrix with entries in* S *given by*

$$M = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,s} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,s} \\ \vdots & \vdots & & \vdots \\ p_{r,1} & p_{r,2} & \cdots & p_{r,s} \end{pmatrix}$$

where each polynomial $p_{i,j} \in S$ *is given by*

$$p_{i,j} = f_{j,1} z_{i,j,1} + f_{j,2} z_{i,j,2} + \cdots + f_{j,m_j} z_{i,j,m_j}.$$

Then

$$\operatorname{grade}\big(I_t(M)\big) \geqslant \min\{r - t + 1, g\}.$$

for $1 \leqslant t \leqslant \min\{r, s\}$.

Proof. Proceed by induction on t. The case $t = 1$ follows from Lemma 5.26 since $I_1(M)$ is generated by the $p_{i,j}$'s themselves.

Now suppose that $1 < t \leqslant \min\{r, s\}$. Let Q be a prime ideal containing $I_t(M)$. If Q contains all the polynomials $p_{i,j}$, then again Lemma 5.26 yields $\operatorname{depth}(S_Q) \geqslant \min\{r, g\} \geqslant \min\{r - t + 1, g\}$. Otherwise, say, $p_{r,s} \notin Q$.

Let M' denote the $(r-1) \times (s-1)$ submatrix of M of the first $r-1$ rows and first $s-1$ columns. Clearly,

$$I_{t-1}\left(M'\right) S_{p_{r,s}} \subset I_t\left(M\right) S_{p_{r,s}}$$

in the localization $S_{\mathfrak{p}_{r,s}}$. The inductive hypothesis gives

$$\begin{aligned}
\operatorname{depth}(S_Q) &\geqslant \operatorname{grade}\left(I_t\left(M\right) S_{\mathfrak{p}_{r,s}}\right) \\
&\geqslant \operatorname{grade}\left(I_{t-1}\left(M'\right) S_{\mathfrak{p}_{r,s}}\right) \\
&\geqslant \operatorname{grade}\left(I_{t-1}\left(M'\right)\right) \\
&\geqslant \min\{(r-1)-(t-1)+1, g\}.
\end{aligned}$$

Therefore, $\operatorname{depth}(S_Q) \geqslant \min\{r-t+1, g\}$ as was to be shown. $\square$

Local cohomology of bigraded algebras

We study the dimension of certain graded parts of local cohomology modules of a finitely generated module over a bigraded algebra. It will come out as a far reaching generalization of Proposition 3.14, a result that has proven to be useful under various situations (see, e.g., proofs of Theorem 3.16 and Theorem 4.8).

The following setup will prevail along this subsection only.

Setup 5.28. *Let $\mathbb{k}$ be a field. Let $\mathfrak{A}$ be a finitely generated standard bigraded $\mathbb{k}$-algebra, i.e. $\mathfrak{A}$ can be generated over $\mathbb{k}$ by the elements of bidegree $(1,0)$ and $(0,1)$. Let R and S be the standard graded algebras over $\mathbb{k}$ given by $R = [\mathfrak{A}]_{(*,0)} = \bigoplus_{j\geqslant 0} [\mathfrak{A}]_{i,0}$ and $S = [\mathfrak{A}]_{(0,*)} = \bigoplus_{k\geqslant 0} [\mathfrak{A}]_{0,k}$, respectively. We set $\mathfrak{m}$ and $\mathfrak{n}$ to be the graded irrelevant ideals of R and S, respectively, that is $\mathfrak{m} = R_+ = \bigoplus_{j>0} [\mathfrak{A}]_{j,0}$ and $\mathfrak{n} = S_+ = \bigoplus_{k>0} [\mathfrak{A}]_{0,k}$.*

Let $\mathbb{M}$ be a bigraded module over $\mathfrak{A}$. Denote by $[\mathbb{M}]_j$ the "one-sided" R-graded part

$$[\mathbb{M}]_j = \bigoplus_{k\in\mathbb{Z}} [\mathbb{M}]_{j,k}.$$

Then, $[\mathbb{M}]_j$ has a natural structured of graded S-module, and its k-th graded part is given by

$$\Big[[\mathbb{M}]_j\Big]_k = [\mathbb{M}]_{j,k}.$$

Note that, for any $i \geqslant 0$, the local cohomology module $H^i_{\mathfrak{m}}(\mathbb{M})$ has a natural structure of bigraded $\mathfrak{A}$-module, and this can be seen from the fact that $H^i_{\mathfrak{m}}(\mathbb{M}) = H^i_{\mathfrak{m}\mathfrak{A}}(\mathbb{M})$ (also, see Lemma 2.2). In particular, each R-graded part

$$\left[H^i_{\mathfrak{m}}(\mathbb{M})\right]_j$$

has a natural structure of graded S-module.

By considering $\mathfrak{A}$ as a "one-sided" graded R-algebra, we get the projective scheme $\operatorname{Proj}_{R\text{-gr}}(\mathfrak{A})$. The sheafification of $\mathbb{M}$, denoted by $\widetilde{\mathbb{M}}$, yields a quasi-coherent $\mathcal{O}_{\operatorname{Proj}_{R\text{-gr}}(\mathfrak{A})}$-module.

For any finitely generated bigraded $\mathfrak{A}$-module $\mathbb{M}$, $\left[H^i_{\mathfrak{m}}(\mathbb{M})\right]_j$ and $H^i\left(\mathrm{Proj}_{R\text{-gr}}(\mathfrak{A}), \widetilde{\mathbb{M}}(j)\right)$ are finitely generated graded S-modules for any $i \geqslant 0, j \in \mathbb{Z}$ (see, e.g., Proposition 3.7, [26, Theorem 2.1]).

The next theorem contains the main result of this subsection.

Theorem 5.29. *Let $\mathbb{M}$ be a finitely generated bigraded $\mathfrak{A}$-module. Then, the following inequalities hold*

(i) $\dim\left(\left[H^i_{\mathfrak{m}}(\mathbb{M})\right]_j\right) \leqslant \min\{\dim(\mathbb{M}) - i, \dim(S)\}$,

(ii) $\dim\left(H^i\left(\mathrm{Proj}_{R\text{-gr}}(\mathfrak{A}), \widetilde{\mathbb{M}}(j)\right)\right) \leqslant \min\{\dim(\mathbb{M}) - i - 1, \dim(S)\}$,

for all $i \geqslant 0, j \in \mathbb{Z}$.

Proof. Let $d = \dim(\mathbb{M})$. Since $\left[H^i_{\mathfrak{m}}(\mathbb{M})\right]_j$ and $H^i\left(\mathrm{Proj}_{R\text{-gr}}(\mathfrak{A}), \widetilde{\mathbb{M}}(j)\right)$ are finitely generated S-modules, it is clear that

$$\dim\left(\left[H^i_{\mathfrak{m}}(\mathbb{M})\right]_j\right) \leqslant \dim(S) \quad \text{and} \quad \dim\left(H^i\left(\mathrm{Proj}_{R\text{-gr}}(\mathfrak{A}), \widetilde{\mathbb{M}}(j)\right)\right) \leqslant \dim(S).$$

(i) By the well-known Grothendieck Vanishing Theorem (see, e.g., [17, Theorem 6.1.2]), $H^i_{\mathfrak{m}}(\mathbb{M}) = 0$ for $i > d$, so that we take $i \leqslant d$.

Proceed by induction on d.

Suppose that $d = 0$. Then $[\mathbb{M}]_{j,k} = 0$ for $k \gg 0$. Since $\left[H^0_{\mathfrak{m}}(\mathbb{M})\right]_j \subset [\mathbb{M}]_j$, we have $\left[\left[H^0_{\mathfrak{m}}(\mathbb{M})\right]_j\right]_k = \left[H^0_{\mathfrak{m}}(\mathbb{M})\right]_{j,k} = 0$ for $k \gg 0$. Thus, $\dim\left(\left[H^0_{\mathfrak{m}}(\mathbb{M})\right]_j\right) = 0$.

Suppose that $d > 0$. There exists a finite filtration

$$0 = \mathbb{M}_0 \subset \mathbb{M}_1 \subset \cdots \subset \mathbb{M}_n = \mathbb{M}$$

of $\mathbb{M}$ such that $\mathbb{M}_l/\mathbb{M}_{l-1} \cong (\mathfrak{A}/\mathfrak{p}_l)(a_l, b_l)$ where $\mathfrak{p}_l \subset \mathfrak{A}$ is a bigraded prime ideal with dimension $\dim(\mathfrak{A}/\mathfrak{p}_l) \leqslant d$ and $a_l, b_l \in \mathbb{Z}$. The short exact sequences

$$0 \to \mathbb{M}_{l-1} \to \mathbb{M}_l \to (\mathfrak{A}/\mathfrak{p}_l)(a_l, b_l) \to 0$$

induce the following long exact sequences in local cohomology

$$\left[H^i_{\mathfrak{m}}(\mathbb{M}_{l-1})\right]_j \to \left[H^i_{\mathfrak{m}}(\mathbb{M}_l)\right]_j \to \left[H^i_{\mathfrak{m}}((\mathfrak{A}/\mathfrak{p}_l)(a_l, b_l))\right]_j.$$

By iterating on l, we get

$$\dim\left(\left[H^i_{\mathfrak{m}}(\mathbb{M})\right]_j\right) \leqslant \max_{1 \leqslant l \leqslant n}\left\{\dim\left(\left[H^i_{\mathfrak{m}}((\mathfrak{A}/\mathfrak{p}_l)(a_l, b_l))\right]_j\right)\right\}.$$

If $\mathfrak{p}_l \supseteq \mathfrak{n}\mathfrak{A}$ then $\mathfrak{A}/\mathfrak{p}_l$ is a quotient of $\mathfrak{A}/\mathfrak{n}\mathfrak{A} \cong R$ and this implies that

$$\left[\left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/\mathfrak{p}_l\right)\right]_j\right]_k = \left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/\mathfrak{p}_l\right)\right]_{j,k} = 0$$

for $k \neq 0$. Thus, we assume that $\mathfrak{p}_l \not\supseteq \mathfrak{n}\mathfrak{A}$.

Alongside with the previous reductions, we can then assume that $\mathbb{M} = \mathfrak{A}/\mathfrak{p}$ where $\mathfrak{p}$ is a bigraded prime ideal and $\mathfrak{p} \not\supseteq \mathfrak{n}\mathfrak{A}$. In this case there exists an homogeneous element $y \in S_1$ such that $y \notin \mathfrak{p}$. The short exact sequence

$$0 \to \left(\mathfrak{A}/\mathfrak{p}\right)(0,-1) \xrightarrow{y} \mathfrak{A}/\mathfrak{p} \to \mathfrak{A}/(y,\mathfrak{p}) \to 0$$

yields the long exact sequence in local cohomology

$$\left[H^{i-1}_{\mathfrak{m}}\left(\mathfrak{A}/(y,\mathfrak{p})\right)\right]_j \to \left(\left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/\mathfrak{p}\right)\right]_j\right)(-1) \xrightarrow{y} \left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/\mathfrak{p}\right)\right]_j \to \left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/(y,\mathfrak{p})\right)\right]_j.$$

Therefore, it follows that

$$\dim\left(\left[H^i_{\mathfrak{m}}(\mathfrak{A}/\mathfrak{p})\right]_j\right) \leqslant \max\left\{\dim\left(\left[H^{i-1}_{\mathfrak{m}}\left(\mathfrak{A}/(y,\mathfrak{p})\right)\right]_j\right), 1+\dim\left(\left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/(y,\mathfrak{p})\right)\right]_j\right)\right\}.$$

Since $\dim\left(\mathfrak{A}/(y,\mathfrak{p})\right) \leqslant d-1$, the induction hypothesis gives

$$\dim\left(\left[H^{i-1}_{\mathfrak{m}}\left(\mathfrak{A}/(y,\mathfrak{p})\right)\right]_j\right) \leqslant (d-1)-(i-1) = d-i$$

and

$$1+\dim\left(\left[H^i_{\mathfrak{m}}\left(\mathfrak{A}/(y,\mathfrak{p})\right)\right]_j\right) \leqslant 1+(d-1)-i = d-i.$$

Therefore, $\dim\left(\left[H^i_{\mathfrak{m}}(\mathfrak{A}/\mathfrak{p})\right]_j\right) \leqslant d-i$, and so the proof of this part follows.

(ii) For $i \geqslant 1$, the isomorphism $H^i\left(\mathrm{Proj}_{R\text{-gr}}(\mathfrak{A}), \widetilde{\mathbb{M}}(j)\right) \cong \left[H^{i+1}_{\mathfrak{m}}(\mathbb{M})\right]_j$ (see, e.g., [87, Corollary 1.5], [47, Theorem A4.1]) and part (i) imply the result. So, $i = 0$ is the only remaining case.

Let $\mathbb{M}' = \mathbb{M}/H^0_{\mathfrak{m}}(\mathbb{M})$ and $\mathfrak{A}' = \mathfrak{A}/\mathrm{Ann}_{\mathfrak{A}}(\mathbb{M}')$. We have the following short exact sequence

$$0 \to \left[\mathbb{M}'\right]_j \to H^0\left(\mathrm{Proj}_{R\text{-gr}}(\mathfrak{A}), \widetilde{\mathbb{M}}(j)\right) \to \left[H^1_{\mathfrak{m}}(\mathbb{M})\right]_j \to 0.$$

(see, e.g., [87, Corollary 1.5], [47, Theorem A4.1]). From part (i), we have the inequality $\dim\left(\left[H^1_{\mathfrak{m}}(\mathbb{M})\right]_j\right) \leqslant \dim(\mathbb{M})-1$.

Therefore, it is enough to show that $\dim\left([\mathbb{M}']_j\right) \leqslant \dim(\mathbb{M})-1$. If $\mathbb{M}' = 0$, the result is clear. Hence, assume that $\mathbb{M}' \neq 0$. Note that $\mathrm{grade}(\mathfrak{m}\mathfrak{A}') > 0$, and so $\dim\left(\mathfrak{A}'/\mathfrak{m}\mathfrak{A}'\right) \leqslant \dim(\mathfrak{A}')-1 \leqslant \dim\left(\mathbb{M}\right)-1$.

Since $\mathfrak{A}' = [\mathfrak{A}']_0 \oplus \mathfrak{m}\mathfrak{A}'$, there is an isomorphism $[\mathfrak{A}']_0 \cong \mathfrak{A}'/\mathfrak{m}\mathfrak{A}'$ of graded $\mathbb{k}$-algebras.

Therefore, the result follows because $[\mathbb{M}']_j$ is a finitely generated module over $[\mathfrak{A}']_0$. $\square$

Of particular interest is the following corollary that generalizes Proposition 3.14.

Corollary 5.30. *Let* $(R, \mathfrak{m})$ *be a standard graded algebra over a field with graded irrelevant ideal* $\mathfrak{m}$*. For any ideal* $I \subset \mathfrak{m}$ *one has*

$$\dim\left(\left[H^i_{\mathfrak{m}}(\mathcal{R}_R(I))\right]_j\right) \leqslant \dim(R) + 1 - i$$

and

$$\dim\left(H^i\left(\operatorname{Proj}_{R\text{-gr}}(\mathcal{R}_R(I)), \mathcal{O}_{\operatorname{Proj}_{R\text{-gr}}(\mathcal{R}_R(I))}(j)\right)\right) \leqslant \dim(R) - i$$

for any $i \geqslant 0, j \in \mathbb{Z}$*.*

Proof. It follows from Theorem 5.29 and the fact that $\dim(\mathcal{R}_R(I)) \leqslant \dim(R)+1$ (Theorem 1.2). $\square$

5.4 Specialization

In this section we study how the process of specializing Rees algebras and saturated fiber cones affects the degree of specialized rational maps, where the latter is understood in terms of coefficient specialization.

The following notation will take over throughout this section.

Setup 5.31. *Essentially keep the basic notation as in the previous section, but this time around take* $A = \mathbb{k}[z_1, \ldots, z_m]$ *to be a polynomial ring over a field* $\mathbb{k}$ *(for the present purpose forget any grading). Consider a rational map* $\mathcal{G} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A$ *given by a representative* $\mathbf{g} = (g_0 : \cdots : g_s)$*, where* $\mathbb{P}^r_A = \operatorname{Proj}(R)$ *with* $R = A[\mathbf{x}] = A[x_0, \ldots, x_r]$*. Fix a maximal ideal* $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ *of* A *where* $\alpha_i \in \mathbb{k}$*.*

Clearly, we have $\mathbb{k} \cong A/\mathfrak{n}$*. We set that the structure of* A*-module of* $\mathbb{k}$ *is given by the canonical homomorphism* $A \twoheadrightarrow A/\mathfrak{n} \cong \mathbb{k}$*. So, we get that*

$$X \times_A \mathbb{k} \cong X \times_A (A/\mathfrak{n}) \qquad \text{and} \qquad M \otimes_A \mathbb{k} \cong M \otimes_A (A/\mathfrak{n}),$$

for any A*-scheme* X *and any* A*-module* M*.*

Then $R/\mathfrak{n}R \cong \mathbb{k}[x_0, \ldots, x_r]$*. Let* $\mathfrak{g}$ *denote the rational map* $\mathfrak{g} : \mathbb{P}^r_{\mathbb{k}} \dashrightarrow \mathbb{P}^s_{\mathbb{k}}$ *with representative* $\overline{\mathbf{g}} = (\overline{g_0} : \cdots : \overline{g_s})$*, where* $\overline{g_i}$ *is the image of* g_i *under the canonical map* $R \twoheadrightarrow R/\mathfrak{n}R$*. Further assume that* $\overline{g_i} \neq 0$ *for all* $0 \leqslant i \leqslant s$*.*

Finally, denote $\mathcal{J} := (g_0, \ldots, g_s) \subset R$ *and* $I := (\mathcal{J}, \mathfrak{n})/\mathfrak{n} = (\overline{g_0}, \ldots, \overline{g_s}) \subset R/\mathfrak{n}R$*.*

Remark 5.32. *Let* X *be a topological space. We say that a general point of* X *has a property* P *if there exists an open dense subset* $U \subset X$ *such that* P *is satisfied by every point of* U*.*

Remark 5.33. *In Setup 5.31, under the assumption of* $\Bbbk$ *being algebraically closed, we will frequently require that a property on* $\Bbbk^m$ *with the Zariski topology holds for a general point. For that, note that we can go back an forth between* $\Bbbk^m$ *and a naturally corresponding subspace of* $\operatorname{Spec}(A)$*, with* $A = \Bbbk[z_1, \ldots, z_m]$*. Namely, consider*

$$\operatorname{MaxSpec}(A) := \left\{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \text{ is maximal}\right\}$$

with the induced Zariski topology of $\operatorname{Spec}(A)$*. Then, by Hilbert's Nullstellensatz, the natural association*

$$(a_1, \ldots, a_m) \mapsto (z_1 - a_1, \ldots, z_m - a_m)$$

yields a homeomorphism of $\Bbbk^m$ *onto* $\operatorname{MaxSpec}(A)$*. In particular, it preserves open dense subsets and an open dense subset in* $\operatorname{Spec}(A)$ *restricts to an open dense subset of* $\operatorname{MaxSpec}(A)$ *naturally identified with an open dense subset of* $\Bbbk^m$*. In particular, if a general point of* $\operatorname{Spec}(A)$ *has a property* P*, then a general point of* $\Bbbk^m$ *has also the property* P*.*

Algebraic lemmata

Lemma 5.34. *With the notation introduced above, we have a commutative diagram*

$$\begin{tikzcd}
A[\mathbf{g}] \arrow[r, hook] \arrow[d, two heads] & \mathcal{R}_R(\mathcal{I}) \arrow[d, two heads] \\
A[\mathbf{g}] \otimes_A \Bbbk \arrow[r, hook] \arrow[d, two heads] & \mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk \arrow[d, two heads] \\
\Bbbk[\overline{\mathbf{g}}] \arrow[r, hook] & \mathcal{R}_{R/\mathfrak{n}R}(I).
\end{tikzcd}$$

where $A[\mathbf{g}]$ *(respectively,* $\Bbbk[\overline{\mathbf{g}}]$*) is identified with* $\mathfrak{F}_R(\mathcal{I})$ *(respectively,* $\mathfrak{F}_{R/\mathfrak{n}R}(I)$*).*

Proof. The upper vertical maps are obvious surjections as $\Bbbk = A/\mathfrak{n}$, hence the upper square is commutative – the lower horizontal map of this square is injective because in the upper horizontal map $A[\mathbf{g}]$ is injected as a direct summand. The right lower vertical map is naturally induced by the natural maps

$$R[t] \twoheadrightarrow R[t] \otimes_A \Bbbk = R[t]/\mathfrak{n}R[t] = A[\mathbf{x}][t]/\mathfrak{n}A[\mathbf{x}][t] \cong A/\mathfrak{n}[\mathbf{x}][t] = \Bbbk[\mathbf{x}][t],$$

where t is a new indeterminate. The left lower vertical map is obtained by restriction thereof. $\square$

Proposition 5.35. *Consider the naturally induced homomorphism of bigraded algebras*

$$\mathfrak{s} : \mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk \twoheadrightarrow \mathcal{R}_{R/\mathfrak{n}R}(I).$$

If I *is not the null ideal, we have:*

(i) $\ker(\mathfrak{s})$ *is a minimal prime ideal of* $\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk$ *and, for any minimal prime* $\mathfrak{Q}$ *of* $\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk$ *other than* $\ker(\mathfrak{s})$*, we have that* $\mathfrak{Q}$ *corresponds with a minimal prime of* $\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk \cong (\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk)/\mathcal{J}(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk)$ *and so*

$$\dim\left((\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk)/\mathfrak{Q}\right) \leqslant \dim(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk).$$

In particular,

$$\begin{aligned}\dim\left(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk\right) &= \max\{\dim(R/\mathfrak{n}R) + 1, \dim(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk)\}\\ &= \max\{r + 2, \dim(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk)\}.\end{aligned}$$

(ii) Let $k \geqslant 0$ *be an integer such that* $\ell(\mathcal{J}_{\mathfrak{P}}) \leqslant \mathrm{ht}(\mathfrak{P}/\mathfrak{n}R) + k$ *for every prime ideal* $\mathfrak{P} \in \mathrm{Spec}(R)$ *containing* $(\mathcal{J}, \mathfrak{n})$*. Then*

$$\dim(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk) \leqslant \dim(R/\mathfrak{n}R) + k.$$

In particular,

$$\dim\left(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk\right) \leqslant \max\{r + 2, r + k + 1\}.$$

(iii) $\dim\left(\ker(\mathfrak{s})\right) \leqslant \dim\left(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk\right)$.

Proof. (i) Let $P \in \mathrm{Spec}(R)$ be a prime ideal not containing $\mathcal{J}$. Localizing the surjection $\mathfrak{s} : \mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk \twoheadrightarrow \mathcal{R}_{R/\mathfrak{n}R}(I)$ at $R \setminus P$, we easily see that it becomes an isomorphism. It follows that some power of $\mathcal{J}$ annihilates $\ker(\mathfrak{s})$, that is

$$\mathcal{J}^l \cdot \ker(\mathfrak{s}) = 0 \tag{5.4}$$

for some $l > 0$. Since $I \neq 0$, then $\mathcal{J} \not\subseteq \ker(\mathfrak{s})$. Therefore, any prime ideal of $\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk$ contains either the prime ideal $\ker(\mathfrak{s})$ or the ideal $\mathcal{J}$. Thus, $\ker(\mathfrak{s})$ is a minimal prime and any other minimal prime $\mathfrak{Q}$ of $\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk$ contains $\mathcal{J}$. Clearly, any such $\mathfrak{Q}$ is a minimal prime of $(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk)/\mathcal{J}(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk) \cong \mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk$. Since $\dim\left(\mathcal{R}_{R/\mathfrak{n}R}(I)\right) = \dim(R/\mathfrak{n}R) + 1$, the claim follows.

(ii) For this, let $\mathfrak{M}$ be a minimal prime of $\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk$ of maximal dimension, i.e.:

$$\dim(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk) = \dim((\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk)/\mathfrak{M}),$$

and let $\mathfrak{P} = \mathfrak{M} \cap R$ be its contraction to R. Clearly, $\mathfrak{P} \supseteq (\mathcal{J}, \mathfrak{n})$. By Lemma 1.24 and the hypothesis,

$$\begin{aligned}
\dim\left(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk\right) &= \dim\big((\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk)/\mathfrak{M}\big)\\
&= \dim(R/\mathfrak{P}) + \mathrm{trdeg}_{R/\mathfrak{P}}\big((\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk)/\mathfrak{M}\big)\\
&= \dim(R/\mathfrak{P}) + \dim\big(((\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \Bbbk)/\mathfrak{M}) \otimes_{R/\mathfrak{P}} R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}\big)\\
&\leqslant \dim(R/\mathfrak{P}) + \dim\big(\mathrm{gr}_{\mathcal{J}}(R) \otimes_R R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}\big)\\
&= \dim(R/\mathfrak{P}) + \ell(\mathcal{J}_{\mathfrak{P}})\\
&\leqslant \dim(R/\mathfrak{P}) + \mathrm{ht}(\mathfrak{P}/\mathfrak{n}) + k\\
&\leqslant \dim(R/\mathfrak{n}R) + k,
\end{aligned}$$

as required.

The supplementary assertion on $\dim\left(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk\right)$ is now clear.

(iii) From (5.4) we have $\mathrm{Ann}_{\mathcal{R}_R(\mathcal{J})\otimes_A \Bbbk}(\ker(\mathfrak{s})) \supseteq \mathcal{J}^l$. Therefore

$$\dim\left(\ker(\mathfrak{s})\right) \leqslant \dim\left(\left(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk\right)/\mathcal{J}\left(\mathcal{R}_R(\mathcal{J}) \otimes_A \Bbbk\right)\right)$$

and so the result follow. $\square$

The next lemma is a consequence of the Primitive Element Theorem and will be useful to study how the degree of rational maps varies under specialization.

Lemma 5.36. *Let $\Bbbk$ denote a field of characteristic zero and let $C \subset B$ stand for a finite extension of finitely generated $\Bbbk$-domains. Let $\mathfrak{b} \subset B$ be a prime ideal and set $\mathfrak{c} := \mathfrak{b} \cap C \subset C$ for its contraction. If C is integrally closed then we have*

$$[\mathrm{Quot}(B/\mathfrak{b}) : \mathrm{Quot}(C/\mathfrak{c})] \leqslant [\mathrm{Quot}(B) : \mathrm{Quot}(C)]\,.$$

Proof. Let $\{b_1, \ldots, b_c\}$ generate B as a C-module. Setting $\overline{C} := C/\mathfrak{c} \subset \overline{B} = B/\mathfrak{b}$, the images $\{\overline{b_1}, \ldots, \overline{b_c}\}$ generate $\overline{B}$ as a $\overline{C}$-module. Since the field extensions $\mathrm{Quot}(B)|\mathrm{Quot}(C)$ and $\mathrm{Quot}(\overline{B})|\mathrm{Quot}(\overline{C})$ are separable, and since $\Bbbk$ is moreover infinite, we can find elements $\lambda_1, \ldots, \lambda_c \in \Bbbk$ such that $L := \sum_{i=1}^{c} \lambda_i b_i \in B$ and $\ell := \sum_{i=1}^{c} \lambda_i \overline{b_i} \in \overline{B}$ are respective primitive elements of the above extensions. Let $X^u + a_1 X^{u-1} + \cdots + a_u = 0$ denote the minimal polynomial of L over $\mathrm{Quot}(C)$. Since C is integrally closed, then $a_i \in C$ for all $1 \leqslant i \leqslant u$ (see, e.g., [110, Theorem 9.2]). Reducing modulo $\mathfrak{b}$, we get $\ell^u + \overline{a_1}\ell^{u-1} + \cdots + \overline{a_u} = 0$. Then the degree of the minimal polynomial of ℓ over $\mathrm{Quot}(\overline{C})$ is at most u, as was to be shown. $\square$

In the following lemma we use the upper semi-continuity of the dimension of fibers.

Lemma 5.37. *The set*

$$\left\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim\left(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A k(\mathfrak{p})\right) \leqslant r+1\right\}$$

is open and dense in $\mathrm{Spec}(A)$.

Proof. We can consider the associated graded ring $\mathrm{gr}_{\mathcal{J}}(R) = \bigoplus_{n \geqslant 0} \mathcal{J}^n/\mathcal{J}^{n+1}$ as a finitely generated single graded A-algebra with zero graded part equal to A. Therefore [47, Theorem 14.8 b.] implies that for every n the subset

$$Z_n = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim\left(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A k(\mathfrak{p})\right) \leqslant n\}$$

is open in $\mathrm{Spec}(A)$.

Let $\mathbb{K} = \mathrm{Quot}(A)$, $\mathbb{T} = R \otimes_A \mathbb{K} = \mathbb{K}[x_0, \ldots, x_r]$ and $\mathbb{I} = \mathcal{J} \otimes_A \mathbb{K}$. The generic fiber of $A \hookrightarrow \mathrm{gr}_{\mathcal{J}}(R)$ is given by $\mathrm{gr}_{\mathcal{J}}(R) \otimes_A \mathbb{K} = \mathrm{gr}_{\mathbb{I}}(\mathbb{T})$. Since $\dim\left(\mathrm{gr}_{\mathbb{I}}(\mathbb{T})\right) = \dim(\mathbb{T}) = r + 1$ (Proposition 1.5), then $(0) \in Z_{r+1}$ and so $Z_{r+1} \neq \emptyset$. $\square$

Geometric picture

One further notation for the geometric environment:

Setup 5.38. *First, recall from Setup 5.31 that* $R = A[x_0, \ldots, x_r]$ *is a standard polynomial ring over* A. *We set* $\mathbb{P}^r_A = \mathrm{Proj}(R)$ *as before. In addition, we had* $A = \mathbb{F}[z_1, \ldots, z_m]$ *and* $\mathfrak{n} \subset A$ *a given (rational) maximal ideal, with* $\Bbbk \cong A/\mathfrak{n}$.

Let $\mathcal{G}$ *and* $\mathfrak{g}$ *be as in Setup 5.31. Denote by* $\mathrm{Proj}(A[\mathbf{g}])$ *and* $\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])$ *the images of* $\mathcal{G}$ *and* $\mathfrak{g}$, *respectively (see Definition-Proposition 5.18). Let* $\mathbb{B}(\mathcal{J}) := \mathrm{BiProj}(\mathcal{R}_R(\mathcal{J}))$ *and* $\mathbb{B}(I) := \mathrm{BiProj}(\mathcal{R}_{R/\mathfrak{n}R}(I))$ *be the graphs of* $\mathcal{G}$ *and* $\mathfrak{g}$, *respectively (see Definition-Proposition 5.22).*

Let $\mathbb{E}(\mathcal{J}) := \mathrm{BiProj}\left(\mathrm{gr}_{\mathcal{J}}(R)\right)$ *be the exceptional divisor of* $\mathbb{B}(\mathcal{J})$.

Consider the commutative diagrams

$$\begin{tikzcd} \mathbb{B}(\mathcal{J}) \arrow[d, "\Pi'"'] \arrow[dr, "\Pi"] & \\ \mathbb{P}^r_A \arrow[r, dashed, "\mathcal{G}"] & \mathrm{Proj}(A[\mathbf{g}]) \end{tikzcd} \tag{5.5}$$

and

$$\begin{tikzcd} \mathbb{B}(I) \arrow[d, "\pi'"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^r_{\Bbbk} \arrow[r, dashed, "\mathfrak{g}"] & \mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}]) \end{tikzcd} \tag{5.6}$$

where Π' and π' are the blowing-up structural maps, which are well-known to be birational (see, e.g., [66, Section II.7]) – note that, had we taken care of a full development of rational/birational maps in the biprojective situation, this fact would be routinely verified.

We see that Π and π fall within the general notion of rational maps with source a biprojective scheme. Most of the presently needed material in the biprojective situation is more or less a straightforward extension of the projective one. Thus, for example, the field of rational functions of the biprojective scheme $\mathbb{B}(\mathcal{J})$ is given by the bihomogeneous localization of $\mathcal{R}_{\mathcal{J}}(\mathcal{J})$ at the null ideal, that is

$$K(\mathbb{B}(\mathcal{J})) := \mathcal{R}_{\mathcal{J}}(\mathcal{J})_{(0)} = \left\{ \frac{f}{g} \mid f, g \in \mathcal{R}_{\mathcal{J}}(\mathcal{J}), \mathrm{bideg}(f) = \mathrm{bideg}(g), g \neq 0 \right\}.$$

Then, the degree of the morphism Π (respectively, Π') is given by

$$[K(\mathbb{B}(\mathcal{J})) : K(\mathrm{Proj}(A[\mathbf{g}]))] \qquad \text{(respectively, } [K(\mathbb{B}(\mathcal{J})) : K(\mathbb{P}^r_A)]\text{)}.$$

Likewise, we have:

Lemma 5.39. *The following statements hold:*

(i) $K(\mathbb{B}(\mathcal{J})) = K(\mathbb{P}^r_{\Bbbk})$.

(ii) Π' *is a birational morphism.*

(iii) $\deg(\Pi) = \deg(\mathcal{G})$.

Proof. (i) It is clear that $K(\mathbb{P}^r_{\Bbbk}) \subset K(\mathbb{B}(\mathcal{J}))$. Let $f/g \in K(\mathbb{B}(\mathcal{J}))$ with $f, g \in [\mathcal{R}_{\mathcal{J}}(\mathcal{J})]_{(\alpha,\beta)}$, then it follows that $f = pt^\beta$ and $g = p't^\beta$ where $p, p' \in [R]_{\alpha+\beta}$. Thus, $f/g = p/p' \in R_{(0)}$ and so $K(\mathbb{B}(\mathcal{J})) \subset K(\mathbb{P}^r_{\Bbbk})$.

(ii) Use essentially the same argument of the implication (B1) $\Rightarrow$ (B2) in Definition-Proposition 5.21. Let η denote the generic point of $\mathbb{B}(\mathcal{J})$ and ξ that of $\mathbb{P}^r_A$. From part (i), $(\Pi')^\sharp_\eta : \mathcal{O}_{\mathbb{P}^r_A,\xi} \to \mathcal{O}_{\mathbb{B}(\mathcal{J}),\eta}$ is an isomorphism. Therefore, [57, Proposition 10.52] yields the existence of dense open subsets $U \subset \mathbb{B}(\mathcal{J})$ and $V \subset \mathbb{P}^r_A$ such that the restriction $\Pi' |_U : U \xrightarrow{\cong} V$ is an isomorphism.

(iii) It follows from part (i). $\square$

Thus, we have as expected: Π and π are generically finite morphisms if and only if $\mathcal{G}$ and $\mathbb{g}$ are so, in which case we have

$$\deg(\mathcal{G}) = \deg(\Pi) \quad \text{and} \quad \deg(\mathbb{g}) = \deg(\pi).$$

Lemma 5.40. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbb{B}(I) \arrow[r, "\pi"] \arrow[d, "p_1"'] & \mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}]) \arrow[d, "q_1"] \\
\mathbb{B}(\mathcal{J}) \times_A \Bbbk \arrow[r, "\Pi \times_A \Bbbk"] \arrow[d, "p_2"'] & \mathrm{Proj}(A[\mathbf{g}]) \times_A \Bbbk \arrow[d, "q_2"] \\
\mathbb{B}(\mathcal{J}) \arrow[r, "\Pi"] & \mathrm{Proj}(A[\mathbf{g}])
\end{tikzcd} \tag{5.7}$$

where the statements below are satisfied:

(i) p_1 *and* q_1 *are closed immersions.*

(ii) p_2 *and* q_2 *are the natural projections from the fiber products.*

Proof. It is an immediate consequence of Lemma 5.34 by taking respective associated Proj and BiProj schemes. $\square$

Corollary 5.41. *The following statements hold:*

(i) $\dim(\mathbb{B}(\mathcal{I})) = \dim(A) + r$.

(ii) $\dim(\mathbb{B}(I)) = r$.

Proof. It follows from Corollary 5.8 and the equalities $\dim(\mathcal{R}_R(\mathcal{I})) = \dim(R) + 1$ and $\dim\left(\mathcal{R}_{R/\mathfrak{n}R}(I)\right) = \dim(R/\mathfrak{n}R) + 1$ (Theorem 1.2). $\square$

The next result is an immediate consequence of Proposition 5.35 and Corollary 5.9.

Lemma 5.42. *Assuming that* $\mathcal{I} \not\subset \mathfrak{n}R$*, the following statements hold:*

(i) $\mathbb{B}(I)$ *is an irreducible component of* $\mathbb{B}(\mathcal{I}) \times_A \mathbb{k}$ *and, for any irreducible component* $\mathcal{Z}$ *of* $\mathbb{B}(\mathcal{I}) \times_A \mathbb{k}$ *other than* $\mathbb{B}(I)$*, we have that* $\mathcal{Z}$ *corresponds with an irreducible component of* $\mathbb{E}(\mathcal{I}) \times_A \mathbb{k}$ *and so*
$$\dim(\mathcal{Z}) \leqslant \dim(\mathbb{E}(\mathcal{I}) \times_A \mathbb{k}).$$

(ii) *Let* $k \geqslant 0$ *be an integer such that* $\ell(\mathcal{I}_{\mathfrak{P}}) \leqslant \mathrm{ht}(\mathfrak{P}/\mathfrak{n}R) + k$ *for every prime ideal* $\mathfrak{P} \in \mathrm{Spec}(R)$ *containing* $(\mathcal{I}, \mathfrak{n})$*. Then*
$$\dim(\mathbb{E}(\mathcal{I}) \times_A \mathbb{k}) \leqslant r + k - 1.$$
In particular,
$$\dim(\mathbb{B}(\mathcal{I}) \times_A \mathbb{k}) \leqslant \max\{r, r + k - 1\}.$$

Main specialization result

Proposition 5.43. *Under Setup 5.38, assume that both* $\mathcal{G}$ *and* $\mathfrak{g}$ *are generically finite. Then, the following statements are satisfied:*

(i) $U = \{y \in \mathrm{Proj}(A[\mathbf{g}]) \mid \Pi^{-1}(y) \text{ is a finite set}\}$ *is an open dense subset in* $\mathrm{Proj}(A[\mathbf{g}])$ *and the restriction* $\Pi^{-1}(U) \to U$ *is a finite morphism.*

(ii) *If* $\dim(\mathbb{E}(\mathcal{I}) \times_A \mathbb{k}) \leqslant r$ *then*
$$q_1^{-1}(q_2^{-1}(U)) \neq \emptyset.$$

Proof. (i) Clearly, Π is a projective morphism, hence is a proper morphism. Thus, as a consequence of Zariski's Main Theorem (see [57, Corollary 12.90]), the subset

$$U = \{y \in \mathrm{Proj}(A[\mathbf{g}]) \mid \Pi^{-1}(y) \text{ is a finite set}\}$$

is open in $\mathrm{Proj}(A[\mathbf{g}])$ and the restriction $\Pi^{-1}(U) \to U$ is a finite morphism. Since Π is generically finite, U is nonempty (see, e.g., [66, Exercise II.3.7]).

(ii) In notation of (5.7), considering $\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])$ as a closed subscheme of $\mathrm{Proj}\,(A[\mathbf{g}]) \times_A \Bbbk$ via q_1, take the restriction

$$\Psi : W = (\Pi \times_A \Bbbk)^{-1}\left(\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])\right) \longrightarrow \mathrm{Proj}\,(\Bbbk[\overline{\mathbf{g}}]).$$

From Lemma 5.42 and the fact that $\mathfrak{g}$ is generically finite, it follows that

$$\dim\left(\mathbb{B}(\mathcal{I}) \times_A \Bbbk\right) = r = \dim\left(\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])\right).$$

Let ξ be the generic point of $\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])$. So the map Ψ is also generically finite, and the fiber $\Psi^{-1}(\xi) = W_\xi = W \times_{\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])} k(\xi)$ of Ψ over ξ is finite.

Letting $w = q_2(q_1(\xi))$, we have the following canonical scheme isomorphisms

$$\begin{aligned}
\Psi^{-1}(\xi) &= W \times_{\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])} k(\xi) \\
&\cong \Big((\mathbb{B}(\mathcal{I}) \times_A \Bbbk) \times_{(\mathrm{Proj}(A[\mathbf{g}]) \times_A \Bbbk)} \mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}]) \Big) \times_{\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])} k(\xi) \\
&\cong (\mathbb{B}(\mathcal{I}) \times_A \Bbbk) \times_{(\mathrm{Proj}(A[\mathbf{g}]) \times_A \Bbbk)} k(\xi) \\
&\cong \Big(\mathbb{B}(\mathcal{I}) \times_{\mathrm{Proj}(A[\mathbf{g}])} (\mathrm{Proj}\,(A[\mathbf{g}]) \times_A \Bbbk) \Big) \times_{(\mathrm{Proj}(A[\mathbf{g}]) \times_A \Bbbk)} k(\xi) \\
&\cong \mathbb{B}(\mathcal{I}) \times_{\mathrm{Proj}(A[\mathbf{g}])} k(\xi) \\
&\cong \left(\mathbb{B}(\mathcal{I}) \times_{\mathrm{Proj}(A[\mathbf{g}])} k(w) \right) \times_{k(w)} k(\xi) \\
&\cong \Pi^{-1}(w) \times_{k(w)} k(\xi),
\end{aligned} \tag{5.8}$$

where $\Pi^{-1}(w) = \mathbb{B}(\mathcal{I})_w = \mathbb{B}(\mathcal{I}) \times_{\mathrm{Proj}(A[\mathbf{g}])} k(w)$ denotes the fiber of Π over w. Thus, it follows that $\dim(\Pi^{-1}(w)) = \dim(\Psi^{-1}(\xi)) = 0$ (see, e.g., [57, Proposition 5.38]) and so $\Pi^{-1}(w)$ is also a finite fiber. Therefore, $w \in U$ and $\xi \in q_1^{-1}(q_2^{-1}(U))$, which clearly implies $q_1^{-1}(q_2^{-1}(U)) \neq \emptyset$. $\square$

Next is the main result of this part.

Theorem 5.44. *Under Setup 5.38, suppose that both $\mathcal{G}$ and $\mathfrak{g}$ are generically finite.*

(i) Assume that the following conditions hold:

(a) $\mathrm{Proj}(A[\mathbf{g}])$ *is a normal scheme.*

(b) $\dim\left(\mathbb{E}(\mathcal{I}) \times_A \Bbbk\right) \leqslant r.$

(c) $\Bbbk$ *is a field of characteristic zero.*

Then

$$\deg(\mathbb{g}) \leqslant \deg(\mathcal{G}).$$

(ii) If $\dim\left(\mathbb{E}(\mathcal{I}) \times_A \Bbbk\right) \leqslant r-1$, *then*

$$\deg(\mathbb{g}) \geqslant \deg(\mathcal{G}).$$

(iii) Assuming that $\Bbbk$ *is algebraically closed, there exists an open dense subset* $\mathcal{W} \subset \Bbbk^m$ *such that, if* $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ *with* $(\alpha_1, \ldots, \alpha_m) \in \mathcal{W}$, *then we have*

$$\deg(\mathbb{g}) \geqslant \deg(\mathcal{G}).$$

(iv) Consider the following condition:
$(\mathbb{K})$ $k \geqslant 0$ *is a given integer such that* $\ell(\mathcal{I}_{\mathfrak{P}}) \leqslant \mathrm{ht}(\mathfrak{P}/\mathfrak{n}R) + k$ *for every prime ideal* $\mathfrak{P} \in \mathrm{Spec}(R)$ *containing* $(\mathcal{I}, \mathfrak{n})$.

Then:

$(\mathbb{K}1)$ *If* $(\mathbb{K})$ *holds with* $k \leqslant 1$, *then condition (b) of part (i) is satisfied.*

$(\mathbb{K}2)$ *If* $(\mathbb{K})$ *holds with* $k = 0$, *then the assumption of (ii) is satisfied.*

Proof. (i) Using condition (b), take an open set U as provided by Proposition 5.43 and shrink it down to an affine open subset $U' := \mathrm{Spec}\,(\mathcal{C}) \subset U$ such that $q_1^{-1}(q_2^{-1}(U')) \neq \emptyset$. The scheme $q_1^{-1}(q_2^{-1}(U'))$ is also affine because q_1 and q_2 are affine morphisms (Lemma 5.40). Then, set

$$q_1^{-1}(q_2^{-1}(U')) =: \mathrm{Spec}\,(C)\,.$$

Since the restriction $\Pi^{-1}(U') \to U'$ is a finite morphism, $\Pi^{-1}(U')$ is also affine (see, e.g., [57, Remark 12.10], [66, Exercise 5.17]). Set $\Pi^{-1}(U') =: \mathrm{Spec}\,(\mathcal{B})$. Similarly,

$$p_1^{-1}(p_2^{-1}(\Pi^{-1}(U'))) =: \mathrm{Spec}\,(B)$$

is also affine.

The following commutative diagram of scheme morphisms stems from these considerations:

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[rr, "\pi\,|_{\mathrm{Spec}(B)}"] \arrow[d] & & \mathrm{Spec}(C) \arrow[d] \\
\mathrm{Spec}(\mathcal{B}) \arrow[rr, "\Pi\,|_{\mathrm{Spec}(\mathcal{B})}"] & & \mathrm{Spec}(\mathcal{C})
\end{tikzcd} \tag{5.9}$$

where $\pi\,|_{\mathrm{Spec}(B)}$ and $\Pi\,|_{\mathrm{Spec}(\mathcal{B})}$ are finite morphisms. It corresponds to the following commutative diagram of ring homomorphisms

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, two heads] & \mathcal{B} \arrow[d, two heads] \\ C \arrow[r, hook] & B \end{tikzcd}$$

with finite horizontal maps, which are injective because $\pi\,|_{\mathrm{Spec}(B)}$ and $\Pi\,|_{\mathrm{Spec}(\mathcal{B})}$ are dominant morphisms and C and $\mathcal{C}$ are integral domains (see [57, Corollary 2.11]). Since $\mathrm{Proj}(A[\mathbf{g}])$ is given to be a normal scheme, $\mathcal{C}$ is integrally closed. By Lemma 5.36,

$$\deg(\mathfrak{g}) = \deg(\pi) = \deg\left(\pi\,|_{\mathrm{Spec}(B)}\right) \leqslant \deg\left(\Pi\,|_{\mathrm{Spec}(\mathcal{B})}\right) = \deg(\Pi) = \deg(\mathcal{G}).$$

(ii) By the hypothesis and Lemma 5.42, we have the set-theoretic equality

$$\mathbb{B}(\mathcal{I}) \times_A \Bbbk = \mathbb{B}(I) \bigcup \mathcal{V}$$

where $\mathcal{V}$ is the union of the irreducible components of $\mathbb{B}(\mathcal{I}) \times_A \Bbbk$ other than $\mathbb{B}(I)$, and

$$\dim(\mathcal{Z}) < r = \dim(\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}]))$$

for each irreducible component $\mathcal{Z} \subset \mathcal{V}$. With notation as in (5.7), considering $\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])$ as a closed subscheme of $\mathrm{Proj}\,(A[\mathbf{g}]) \times_A \Bbbk$ via q_1, take the restriction

$$\Psi : W = (\Pi \times_A \Bbbk)^{-1}\,(\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])) \longrightarrow \mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}]).$$

Let ξ be the generic point of $\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])$ and denote $w = q_2(q_1(\xi))$. If $\mathcal{Z}$ is any irreducible component of $\mathbb{B}(\mathcal{I}) \times_A \Bbbk$ other than $\mathbb{B}(I)$, we have $\Psi^{-1}(\xi) \cap \mathcal{Z} = \emptyset$, since otherwise the restriction

$$\Psi\,|_{(W \cap \mathcal{Z})} : (W \cap \mathcal{Z}) \to \mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}])$$

gives a dominant morphism, thus implying that $\dim(\mathcal{Z}) \geqslant \dim(\mathrm{Proj}(\Bbbk[\overline{\mathbf{g}}]))$, which is a contradiction. Therefore, $\Psi^{-1}(\xi) \subset \mathbb{B}(I)$, and so $\Psi^{-1}(\xi)$ and $\pi^{-1}(\xi)$ have the same cardinality. Since π is assumed to be generically finite, the generic point u of $\mathbb{B}(I)$ is the only point of $\pi^{-1}(\xi)$. Thus, set-theoretically $\pi^{-1}(\xi) = \{u\}$ and $\Psi^{-1}(\xi) = \{p_1(u)\}$.

Referring to (5.9), we take the affine open subsets $\mathrm{Spec}(D) := p_2^{-1}\,(\mathrm{Spec}(\mathcal{B})) \subset \mathbb{B}(\mathcal{I}) \times_A \Bbbk$ and $\mathrm{Spec}(E) := q_2^{-1}\,(\mathrm{Spec}(\mathcal{C})) \subset \mathrm{Proj}(A[\mathbf{g}]) \times_A \Bbbk$. Then, there is an induced commutative diagram of scheme morphisms

$$\begin{tikzcd}
\operatorname{Spec}(B) \arrow[r, "\pi|_{\operatorname{Spec}(B)}"] \arrow[d] & \operatorname{Spec}(C) \arrow[d] \\
\operatorname{Spec}(D) \arrow[r, "(\Pi \times_A \Bbbk)|_{\operatorname{Spec}(D)}"] & \operatorname{Spec}(E)
\end{tikzcd}$$

with corresponding commutative diagram of ring homomorphisms

$$\begin{tikzcd}
E \arrow[r] \arrow[d, two heads] & D \arrow[d, two heads] \\
C \arrow[r, hook] & B
\end{tikzcd}$$

where B and C are integral domains, while D and E may not be. Also, the homomorphism $E \to D$ is not necessarily injective (see [57, Corollary 2.11]).

From (5.4), we obtain $\mathcal{I}^l \cdot \ker(\mathfrak{s}) = 0$ where $\mathfrak{s} : \mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk \twoheadrightarrow \mathcal{R}_{R/\mathfrak{n}R}(I)$. Since $I \neq 0$, it follows that $\mathcal{I} \not\subseteq \ker(\mathfrak{s})$ and so $\ker(\mathfrak{s}) \notin V(\mathcal{I} \cdot (\mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk)) \supset \operatorname{Supp}_{\mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk}(\ker(\mathfrak{s}))$. In terms of sheaves, the closed immersion p_1 in (5.7) gives the short exact sequence

$$0 \to \mathfrak{J} \to \mathcal{O}_{\mathbb{B}(\mathcal{I}) \times_A \Bbbk} \to p_{1*}\mathcal{O}_{\mathbb{B}(I)} \to 0 \tag{5.10}$$

where $\mathfrak{J}$ is the sheafification of the ideal $\ker(\mathfrak{s})$. Then, it follows that $p_1(u) \notin \operatorname{Supp}(\mathfrak{J})$. Restricting (5.10) to $\operatorname{Spec}(D)$ yields the exact sequence

$$0 \to \mathfrak{Q} \to D \to B \to 0 \tag{5.11}$$

where $\mathfrak{Q}$ is the ideal associated with the restriction $\mathfrak{J}|_{\operatorname{Spec}(D)}$. Since $B \cong D/\mathfrak{Q}$, the ideal $\mathfrak{Q} \in \operatorname{Spec}(D)$ is the prime ideal of the point $p_1(u)$, and therefore $\mathfrak{Q}$ is not in the support of $\mathfrak{Q}$ as a D-module (i.e., $\mathfrak{Q} \notin \operatorname{Supp}_D(\mathfrak{Q})$).

Now, after these reductions we have $\Psi^{-1}(\xi) \cong \operatorname{Spec}(D \otimes_E \operatorname{Quot}(C))$ and $\pi^{-1}(\xi) \cong \operatorname{Spec}(B \otimes_C \operatorname{Quot}(C))$ as schemes. Since $E \twoheadrightarrow C$ is surjective, $C \cong E/J$ for some ideal $J \subset E$. Since B is a C-module, then $B \otimes_E C \cong B/JB = B$ and $JD \subset \mathfrak{Q}$. By applying the tensor product $- \otimes_E C$ to (5.11) we get the exact sequence

$$0 \to \mathfrak{Q}/JD \to D/JD \to B \to 0. \tag{5.12}$$

We also have that $\mathfrak{Q}/JD \notin \operatorname{Supp}_{D/JD}(\mathfrak{Q}/JD)$. From the fact that $B \otimes_C \operatorname{Quot}(C) = \operatorname{Quot}(B) \neq 0$, then $(D/JD) \otimes_C \operatorname{Quot}(C) \neq 0$ and so we have an injection $C \hookrightarrow D/JD$. Tensoring (5.12) with $\operatorname{Quot}(C)$ over C, we obtain the exact sequence

$$0 \to \mathfrak{q} \to (D/JD) \otimes_C \operatorname{Quot}(C) \to B \otimes_C \operatorname{Quot}(C) \to 0$$

where $\mathfrak{q} = (\mathfrak{Q}/\mathcal{J}D) \otimes_C \mathrm{Quot}(C)$ and $\mathfrak{q} \notin \mathrm{Supp}_{(D/\mathcal{J}D)\otimes_C \mathrm{Quot}(C)}(\mathfrak{q})$. Since $\Psi^{-1}(\xi)$ has only one point then $\mathfrak{q}$ is the unique prime ideal of $(D/\mathcal{J}D) \otimes_C \mathrm{Quot}(C) \cong D \otimes_E \mathrm{Quot}(C)$, and this necessarily implies that $\mathfrak{q} = (0)$.

Therefore, there is actually an isomorphism $\Psi^{-1}(\xi) \cong \pi^{-1}(\xi)$ of schemes.

By (5.8), $\Psi^{-1}(\xi) \cong \Pi^{-1}(w) \times_{k(w)} k(\xi)$, from which follows that

$$\dim_{k(\xi)}\left(\mathcal{O}(\Psi^{-1}(\xi))\right) = \dim_{k(w)}\left(\mathcal{O}(\Pi^{-1}(w))\right).$$

Let U be the open set of Proposition 5.43 and η be the generic point of $\mathrm{Proj}(A[\mathbf{g}])$. Consider the finite morphism

$$\Pi^{-1}(U) \to U.$$

Then we have

$$\begin{aligned}\deg(\mathbb{g}) &= \dim_{k(\xi)}\left(\mathcal{O}(\pi^{-1}(\xi))\right)\\ &= \dim_{k(\xi)}\left(\mathcal{O}(\Psi^{-1}(\xi))\right)\\ &= \dim_{k(w)}\left(\mathcal{O}(\Pi^{-1}(w))\right)\\ &\geqslant \dim_{k(\eta)}\left(\mathcal{O}(\Pi^{-1}(\eta))\right) = \deg(\mathcal{G}),\end{aligned}$$

where the inequality follows by the upper semi-continuity of the degree of the fibers of a dominant finite morphism between integral schemes (see, e.g., [106, Exercise 5.1.25], [57, Corollary 7.30]).

(iii) By setting

$$\mathcal{W} = \left\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim\left(\mathrm{gr}_{\mathcal{J}}(R) \otimes_A k(\mathfrak{p})\right) \leqslant r+1\right\} \cap \mathrm{MaxSpec}(A),$$

the result is obtained from Remark 5.33, Lemma 5.37, Corollary 5.9 and part (ii).

(iv) Both $(\mathbb{K}1)$ and $(\mathbb{K}2)$ follow from Lemma 5.42. $\square$

Specialization of the saturated fiber cone

This subsection deals with the problem of specializing saturated fiber cones. Under certain general conditions it will turn out that the multiplicity of the saturated fiber cone remains constant under specialization.

The reader is referred to the notation of Section 5.2.

Setup 5.45. *Keep the notation introduced in Setup 5.31 and Setup 5.38. Let* $\mathbb{K} = \mathrm{Quot}(A)$ *denote the field of fractions of* A *and let* $\mathbb{T} := \mathbb{K}[x_0, \ldots, x_r]$ *denote the standard polynomial ring over* $\mathbb{K}$ *obtained from* $R = A[x_0, \ldots, x_r]$ *by base change (i.e., considering the* A*-coefficients of a polynomial as* $\mathbb{K}$*-coefficients). Let* $\mathcal{G}$ *and* $\mathbb{g}$ *be as in Setup 5.31.*

In addition, let $\mathbb{G}$ *denote the rational map* $\mathbb{G} : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow \mathbb{P}^s_{\mathbb{K}}$ *with representative* $\mathbf{G} = (G_0 : \cdots : G_s)$*, where* G_i *is the image of* g_i *along the canonical inclusion* $R \hookrightarrow \mathbb{T}$*. Set* $\mathbb{I} := (G_0, \ldots, G_s) \subset \mathbb{T}$*.*

Finally, let $Y := \operatorname{Proj}(\mathbb{k}[\bar{\mathbf{g}}])$ *and* $\mathbb{Y} := \operatorname{Proj}(\mathbb{K}[\mathbf{G}])$ *be the images of* $\mathfrak{g}$ *and* $\mathbb{G}$*, respectively (see Definition-Proposition 5.18).*

As in Remark 5.24, the rational map $\mathcal{G}$ is generically finite if and only if the rational map $\mathbb{G}$ is so, and we have the equality $\deg(\mathcal{G}) = \deg(\mathbb{G})$.

Consider the projective R-scheme $\operatorname{Proj}_{R\text{-gr}}(\mathcal{R}_R(\mathcal{I}))$, where $\mathcal{R}_R(\mathcal{I})$ is viewed as a "one-sided" R-graded algebra.

For any $\mathfrak{p} \in \operatorname{Spec}(A)$, let $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. The fiber $\mathcal{R}_R(\mathcal{I}) \otimes_A k(\mathfrak{p})$ inherits a one-sided structure of a graded $R(\mathfrak{p})$-algebra, where $R(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = k(\mathfrak{p})[x_0, \ldots, x_r]$. Moreover, it has a natural structure as a bigraded algebra over $R(\mathfrak{p})[y_0, \ldots, y_s] = R(\mathfrak{p}) \otimes_A A[y_0, \ldots, y_s]$.

Therefore, for $0 \leqslant i \leqslant r$ the sheaf cohomology

$$\mathcal{M}(\mathfrak{p})^i := \mathrm{H}^i\left(\operatorname{Proj}_{R(\mathfrak{p})\text{-gr}}\left(\mathcal{R}_R(\mathcal{I}) \otimes_A k(\mathfrak{p})\right), \mathcal{O}_{\operatorname{Proj}_{R(\mathfrak{p})\text{-gr}}(\mathcal{R}_R(\mathcal{I})\otimes_A k(\mathfrak{p}))}\right) \tag{5.13}$$

has a natural structure as a finitely generated graded $k(\mathfrak{p})[y_0, \ldots, y_s]$-module (see, e.g., Proposition 3.7). In particular, we can consider its Hilbert function $\mathrm{H}\left(\mathcal{M}(\mathfrak{p})^i, t\right) := \dim_{k(\mathfrak{p})}\left(\left[\mathcal{M}(\mathfrak{p})^i\right]_t\right)$.

Lemma 5.46. *For a given* $\mathfrak{p} \in \operatorname{Spec}(A)$*, consider the function* $\chi_\mathfrak{p} : \mathbb{N} \to \mathbb{N}$ *defined by*

$$\chi_\mathfrak{p}(t) := \sum_{i=0}^{r} (-1)^i \mathrm{H}\left(\mathcal{M}(\mathfrak{p})^i, t\right).$$

Then, there exists an open dense subset $\mathcal{U} \subset \operatorname{Spec}(A)$*, such that* $\chi_\mathfrak{p}$ *is the same for all* $\mathfrak{p} \in \mathcal{U}$*.*

Proof. Consider the affine open covering

$$\mathcal{W} := \left(\operatorname{Spec}\left(\mathcal{R}_R(\mathcal{I})_{(x_i)}\right)\right)_{0 \leqslant i \leqslant r}$$

of $\operatorname{Proj}_{R\text{-gr}}(\mathcal{R}_R(\mathcal{I}))$, with corresponding Čech complex

$$C^\bullet(\mathcal{W}): \quad 0 \to \bigoplus_i \mathcal{R}_R(\mathcal{I})_{(x_i)} \to \bigoplus_{i<j} \mathcal{R}_R(\mathcal{I})_{(x_i x_j)} \to \cdots \to \mathcal{R}_R(\mathcal{I})_{(x_0 \cdots x_r)} \to 0.$$

Note that each $C^i(\mathcal{W})$ has a natural structure of finitely generated graded algebra over A, and its grading comes from the graded structure of $A[\mathbf{y}]$. By using the Generic Freeness Lemma (see, e.g., [47, Theorem 14.4]), there exist elements $a_i \in A$ such that each graded component of the localization $C^i(\mathcal{W})_{a_i}$ is a free module over A_{a_i}.

Let $D^\bullet$ be the complex given by $D^i = C^i(\mathcal{W})_a$, where $a = a_0 a_1 \cdots a_r$. Hence, now $D^\bullet$ is a complex of graded $A_a[\mathbf{y}]$-modules and each graded strand $[D^\bullet]_t$ is a complex of free A_a-modules. Notice that each of the free A_a-modules $\left[D^i\right]_t$ is almost never finitely generated.

The i-th cohomology of a (co-)complex $F^\bullet$ is denoted by $H^i(F^\bullet)$. Since each $[D^\bullet]_t$ is a complex of free A_a-modules (in particular, flat), [66, Lemma III.12.3] yields the existence of complexes $L_t^\bullet$ of finitely generated free A_a-modules such that

$$H^i\Big([D^\bullet]_t \otimes_{A_a} k(\mathfrak{p})\Big) \cong H^i\Big(L_t^\bullet \otimes_{A_a} k(\mathfrak{p})\Big) \tag{5.14}$$

for all $\mathfrak{p} \in \text{Spec}(A_a) \subset \text{Spec}(A)$. Let $\mathcal{U} := \text{Spec}(A_a) \subset \text{Spec}(A)$.

Claim. $\chi_\mathfrak{p}$ is independent of $\mathfrak{p}$ on $\mathcal{U}$; in other words, for any $\mathfrak{p} \in \mathcal{U}$ and any $\mathfrak{q} \in \mathcal{U}$, we have $\chi_\mathfrak{p}(t) = \chi_\mathfrak{q}(t)$ for every $t \in \mathbb{N}$.

Consider an arbitrary $\mathfrak{p} \in \mathcal{U}$. Since $\mathcal{R}_R(\mathcal{J}) \otimes_A k(\mathfrak{p}) \cong \mathcal{R}_R(\mathcal{J})_a \otimes_{A_a} k(\mathfrak{p})$, then $D^\bullet \otimes_{A_a} k(\mathfrak{p})$ coincides with the Čech complex corresponding with the affine open covering

$$\Big(\text{Spec}\,\Big(\big(\mathcal{R}_R(\mathcal{J}) \otimes_A k(\mathfrak{p})\big)_{(x_i)}\Big)\Big)_{0\leqslant i\leqslant r}$$

of $\text{Proj}_{R(\mathfrak{p})\text{-gr}}\,(\mathcal{R}_R(\mathcal{J}) \otimes_A k(\mathfrak{p}))$. Hence, from (5.13) and (5.14), for any $t \in \mathbb{N}$ there is an isomorphism

$$\Big[\mathcal{M}(\mathfrak{p})^i\Big]_t \cong H^i\Big(L_t^\bullet \otimes_{A_a} k(\mathfrak{p})\Big).$$

But since each L_t^i is a finitely generated free A_a-module, it follows that

$$\sum_{i=0}^{r} (-1)^i \dim_{k(\mathfrak{p})}\left(\Big[\mathcal{M}(\mathfrak{p})^i\Big]_t\right) = \sum_{i=0}^{r} (-1)^i \text{rank}_{A_a}\left(L_t^i\right).$$

Therefore, for every $t \in \mathbb{N}$, $\chi_\mathfrak{p}(t) = \sum_{i=0}^{r}(-1)^i H\left(\mathcal{M}(\mathfrak{p})^i, t\right)$ does not depend on $\mathfrak{p}$. □

The following theorem contains the main result of this part. By considering saturated fiber cones, we ask how the product of the degrees of the map and of its image behave under specialization.

Theorem 5.47. *Under Setup 5.45, suppose that both $\mathcal{G}$ and $\mathbb{g}$ are generically finite. Assuming that $\mathbb{k}$ is algebraically closed, there exists an open dense subset $\mathcal{V} \subset \mathbb{k}^m$ such that, if $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ with $(\alpha_1, \ldots, \alpha_m) \in \mathcal{V}$, then we have*

$$\deg(\mathbb{g}) \cdot \deg_{\mathbb{P}^s_{\mathbb{k}}}(Y) = e\left(\widetilde{\mathfrak{F}_{R/\mathfrak{n}R}(I)}\right) = e\left(\widetilde{\mathfrak{F}_{\mathbb{T}}(\mathbb{I})}\right) = \deg(\mathbb{G}) \cdot \deg_{\mathbb{P}^s_{\mathbb{K}}}(\mathbb{Y}).$$

Proof. Take an open dense subset $\mathcal{U}$ like the one of Lemma 5.46, then from Lemma 5.37 we have that

$$\mathcal{V} = \mathcal{U} \cap \left\{\mathfrak{p} \in \text{Spec}(A) \mid \dim\left(\text{gr}_{\mathcal{J}}(R) \otimes_A k(\mathfrak{p})\right) \leqslant r+1\right\} \cap \text{MaxSpec}(A)$$

is an open dense subset of $\text{MaxSpec}(A)$. From Remark 5.33, $\mathcal{V}$ induces an open dense subset in $\mathbb{k}^m$.

From now on, suppose that $\mathfrak{n} \in \mathcal{V}$.

Let $W := \mathrm{Proj}_{R(\mathfrak{n})\text{-gr}}(\mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk)$, as in (5.13), $H^i(W, \mathcal{O}_W) = \mathcal{M}(\mathfrak{n})^i$. By a similar token, $H^i(\mathbb{W}, \mathcal{O}_{\mathbb{W}}) = \mathcal{M}((0))^i$, where $\mathbb{W} := \mathrm{Proj}_{R(0)\text{-gr}}(\mathcal{R}_R(\mathcal{I}) \otimes_A \mathbb{K})$, with $R(0) := R \otimes_A \mathbb{K} = \mathbb{K}[x_0, \ldots, x_r]$ and (0) denotes the null ideal of A.

Now, clearly $(0) \in \mathcal{V}$ and $\mathfrak{n} \in \mathcal{V}$. Therefore, Lemma 5.46 yields the equalities

$$\sum_{i=0}^{r} (-1)^i H\Big(H^i(W, \mathcal{O}_W), t\Big) = \sum_{i=0}^{r} (-1)^i H\Big(H^i(\mathbb{W}, \mathcal{O}_{\mathbb{W}}), t\Big) \tag{5.15}$$

for all $t \in \mathbb{N}$.

From the definition of $\mathcal{V}$ and Proposition 5.35 we get that $\dim(\mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk) = \dim(R/\mathfrak{n}R) + 1$. Hence, for any $i \geqslant 1$, Theorem 5.29(ii) implies the inequalities

$$\dim\left(H^i(W, \mathcal{O}_W)\right) \leqslant \dim(R/\mathfrak{n}R) - 1 \;\text{ and }\; \dim\left(H^i(\mathbb{W}, \mathcal{O}_{\mathbb{W}})\right) \leqslant \dim(\mathbb{T}) - 1.$$

Therefore, (5.15) gives that

$$\dim\left(H^0(W, \mathcal{O}_W)\right) = \dim\left(H^0(\mathbb{W}, \mathcal{O}_{\mathbb{W}})\right) = \dim(\mathbb{T}) = \dim(R/\mathfrak{n}R),$$

and that the leading coefficients of the Hilbert polynomials of $H^0(W, \mathcal{O}_W)$ and $H^0(\mathbb{W}, \mathcal{O}_{\mathbb{W}})$ coincide, and so $e\left(H^0(W, \mathcal{O}_W)\right) = e\left(H^0(\mathbb{W}, \mathcal{O}_{\mathbb{W}})\right)$.

Consider the exact sequence of finitely generated graded $(\mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk)$-modules

$$0 \to \ker(\mathfrak{s}) \to \mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk \xrightarrow{\mathfrak{s}} \mathcal{R}_{R/\mathfrak{n}R}(I) \to 0$$

where $\mathfrak{s} : \mathcal{R}_R(\mathcal{I}) \otimes_A \Bbbk \twoheadrightarrow \mathcal{R}_{R/\mathfrak{n}R}(I)$ denotes the same canonical map of Proposition 5.35.

Sheafifying and taking the long exact sequence in cohomology yield an exact sequence of finitely generated graded $\Bbbk[\mathbf{y}]$-modules

$$0 \to H^0\left(W, \ker(\mathfrak{s})^\sim\right) \to H^0\left(W, \mathcal{O}_W\right) \to H^0\left(W, \mathcal{R}_{R/\mathfrak{n}R}(I)^\sim\right) \to H^1\left(W, \ker(\mathfrak{s})^\sim\right).$$

Note that

$$\begin{aligned}\widetilde{\mathfrak{F}_{R/\mathfrak{n}R}(I)} &\cong H^0\left(\mathrm{Proj}_{(R/\mathfrak{n}R)\text{-gr}}\left(\mathcal{R}_{R/\mathfrak{n}R}(I)\right), \mathcal{O}_{\mathrm{Proj}_{(R/\mathfrak{n}R)\text{-gr}}\left(\mathcal{R}_{R/\mathfrak{n}R}(I)\right)}\right) \\ &\cong H^0\left(W, \mathcal{R}_{R/\mathfrak{n}R}(I)^\sim\right)\end{aligned}$$

(see, e.g., [66, Lemma III.2.10]).

From the definition of $\mathcal{V}$ and Proposition 5.35, it follows that $\dim(\ker(\mathfrak{s})) \leqslant \dim(R/\mathfrak{n}R)$. Hence Theorem 5.29 gives that

$$\dim\left(H^0\left(W, \ker(\mathfrak{s})^\sim\right)\right) \leqslant \dim(R/\mathfrak{n}R) - 1 \;\text{ and }\; \dim\left(H^1\left(W, \ker(\mathfrak{s})^\sim\right)\right) \leqslant \dim(R/\mathfrak{n}R) - 2.$$

Therefore, we get the equality

$$e\left(\widetilde{\mathfrak{F}_{R/\mathfrak{n}R}(I)}\right) = e\left(H^0\left(W, \mathcal{O}_W\right)\right).$$

Since $\widetilde{\mathfrak{F}_{\mathbb{T}}(\mathbb{I})} \cong H^0(\mathbb{W}, \mathcal{O}_{\mathbb{W}})$, summing up yields

$$e\left(\widetilde{\mathfrak{F}_{R/\mathfrak{n}R}(I)}\right) = e\left(H^0\left(W, \mathcal{O}_W\right)\right) = e\left(H^0\left(\mathbb{W}, \mathcal{O}_{\mathbb{W}}\right)\right) = e\left(\widetilde{\mathfrak{F}_{\mathbb{T}}(\mathbb{I})}\right).$$

Finally, by Theorem 3.4 it follows that

$$e\left(\widetilde{\mathfrak{F}_{R/\mathfrak{n}R}(I)}\right) = \deg(\mathfrak{g}) \cdot \deg_{\mathbb{P}^s_{\Bbbk}}(Y) \quad \text{and} \quad e\left(\widetilde{\mathfrak{F}_{\mathbb{T}}(\mathbb{I})}\right) = \deg(\mathbb{G}) \cdot \deg_{\mathbb{P}^s_{\mathbb{K}}}(\mathbb{Y}),$$

and so the result is obtained. □

Now as an easy corollary we show that under general conditions the degree of the image of a rational map never increases under specialization.

Corollary 5.48. *Under Setup 5.45, suppose that both $\mathcal{G}$ and $\mathfrak{g}$ are generically finite. Assuming that $\Bbbk$ is algebraically closed, there exists an open dense subset $\Omega \subset \Bbbk^m$ such that, if $\mathfrak{n} = (z_1 - \alpha_1, \ldots, z_m - \alpha_m)$ with $(\alpha_1, \ldots, \alpha_m) \in \Omega$, then we have*

$$\deg_{\mathbb{P}^s_{\Bbbk}}(Y) \leqslant \deg_{\mathbb{P}^s_{\mathbb{K}}}(\mathbb{Y}).$$

Proof. It follows from Theorem 5.44(iii) and Theorem 5.47. □

5.5 Perfect ideals of height two

In this section we deal with the case of a rational map $\mathcal{F} : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^r_{\Bbbk}$ with a perfect base ideal of height 2, where $\Bbbk$ is a field of characteristic zero. Note that the condition G_{r+1} is satisfied for the generic perfect ideal of height 2.

The main idea is that we can compute the degree of the rational map (Corollary 4.13) when the condition G_{r+1} is satisfied, then a suitable application of Theorem 5.44 gives an upper bound for all the rational maps that satisfy the weaker condition F_0.

Below Setup 5.31 is adapted to the particular case of a perfect ideal of height 2.

Notation 5.49. *Let $\Bbbk$ be a field of characteristic zero. Let $1 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_r$ be integers with $\mu_1 + \mu_2 + \cdots + \mu_r = d$. Given integers $1 \leqslant i \leqslant r+1$ and $1 \leqslant j \leqslant r$, let*

$$\mathbf{z}_{i,j} = \{z_{i,j,1}, z_{i,j,2}, \ldots, z_{i,j,m_j}\}$$

denote a set of variables over $\mathbb{F}$, of cardinality $m_j := \binom{\mu_j + r}{r}$ – the number of coefficients of a polynomial of degree μ_j in $r+1$ variables.

Let $\mathbf{z}$ *be the set of mutually independent variables* $\mathbf{z} = \bigcup_{i,j} \mathbf{z}_{i,j}$, A *be the polynomial ring* $A = \Bbbk[\mathbf{z}]$, *and* R *be the polynomial ring* $R = A[x_0, \ldots, x_r]$. *Let* $\mathcal{M}$ *be the* $(r+1) \times r$ *matrix with entries in* R *given by*

$$\mathcal{M} = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,r} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,r} \\ \vdots & \vdots & & \vdots \\ p_{r+1,1} & p_{r+1,2} & \cdots & p_{r+1,r} \end{pmatrix}$$

where each polynomial $p_{i,j} \in R$ *is given by*

$$p_{i,j} = z_{i,j,1}x_0^{\mu_j} + z_{i,j,2}x_0^{\mu_j-1}x_1 + \cdots + z_{i,j,m_j-1}x_{r-1}x_r^{\mu_j-1} + z_{i,j,m_j}x_r^{\mu_j}.$$

Fix a (rational) maximal ideal $\mathfrak{n} := (z_{i,j,k} - \alpha_{i,j,k}) \subset A$ *of* A, *with* $\alpha_{i,j,k} \in \Bbbk$.

Set $\mathbb{K} := \mathrm{Quot}(A)$, *and denote* $\mathbb{T} := R \otimes_A \mathbb{K} = \mathbb{K}[x_0, \ldots, x_r]$ *and* $R/\mathfrak{n}R \cong \Bbbk[x_0, \ldots, x_r]$.

Let $\mathbb{M}$ *and* M *denote respectively the matrix* $\mathcal{M}$ *viewed as a matrix with entries over* $\mathbb{T}$ *and* $R/\mathfrak{n}R$. *Let* $\{g_0, g_1, \ldots, g_r\} \subset R$ *be the ordered signed minors of the matrix* $\mathcal{M}$. *Then, the ordered signed minors of* $\mathbb{M}$ *and* M *are given by* $\{G_0, G_1, \ldots, G_r\} \subset \mathbb{T}$ *and* $\{\overline{g_0}, \overline{g_1}, \ldots, \overline{g_r}\} \subset R/\mathfrak{n}R$, *respectively, where* $G_i = g_i \otimes_R \mathbb{T}$ *and* $\overline{g_i} = g_i \otimes_R (R/\mathfrak{n}R)$.

Let $\mathcal{G} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^r_A$, $\mathbb{G} : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow \mathbb{P}^r_{\mathbb{K}}$ *and* $\mathbb{g} : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^r_{\Bbbk}$ *be the rational maps given by the representatives* $(g_0 : \cdots : g_r)$, $(G_0 : \cdots : G_r)$ *and* $(\overline{g_0} : \cdots : \overline{g_r})$, *respectively.*

Lemma 5.50. *The following statements hold:*

(i) The ideal $I_r(\mathbb{M})$ *is perfect of height two and satisfies the condition* G_{r+1}.

(ii) The rational map $\mathbb{G} : \mathbb{P}^r_{\mathbb{K}} \dashrightarrow \mathbb{P}^r_{\mathbb{K}}$ *is generically finite.*

Proof. Let $\mathbb{I} = I_r(\mathbb{M})$.

(i) The claim that $\mathbb{I}$ is perfect of height two is clear from Hilbert-Burch Theorem (see, e.g., [47, Theorem 20.15]).

From Proposition 5.27, $\mathrm{ht}(I_i(\mathbb{M})) \geqslant \mathrm{ht}(I_i(\mathcal{M})) \geqslant r + 2 - i$ for $1 \leqslant i \leqslant r$. Since the G_{r+1} condition on $\mathbb{I}$ (see Definition 1.33) is equivalent to

$$\mathrm{ht}(I_{r+1-i}(\mathbb{M})) = \mathrm{ht}(\mathrm{Fitt}_i(\mathbb{I})) > i$$

for $1 \leqslant i \leqslant r$, and so the result follows.

(ii) Since $\mu(\mathbb{I}) = r + 1 = \dim(\mathbb{T})$, then the condition G_{r+1} is equivalent to G_∞. Thus, we get from Theorem 1.37 that $\mathbb{I}$ is of linear type and so $\ell(\mathbb{I}) = \dim(\mathfrak{F}_{\mathbb{T}}(\mathbb{I})) = r + 1$.

Finally, since $\mathfrak{F}_{\mathbb{T}}(\mathbb{I})$ corresponds with the homogeneous coordinate ring of the image of $\mathbb{G}$, then the result follows. ☐

The main result of this section is a straightforward application of the previous developments.

Theorem 5.51. *Let $\mathbb{k}$ be a field of characteristic zero and let $D = \mathbb{k}[x_0, \ldots, x_r]$ denote a polynomial ring over $\mathbb{k}$. Let $I \subset D$ be a perfect ideal of height two minimally generated by $r+1$ forms $\{f_0, f_1, \ldots, f_r\}$ of the same degree d and Hilbert-Burch resolution of the form*

$$0 \to \bigoplus_{i=1}^{r} D(-d-\mu_i) \xrightarrow{\varphi} D(-d)^{r+1} \to I \to 0.$$

Consider the rational map $\mathcal{F} : \mathbb{P}^r_{\mathbb{k}} \dashrightarrow \mathbb{P}^r_{\mathbb{k}}$ given by

$$(x_0 : \cdots : x_r) \mapsto \big(f_0(x_0, \ldots, x_r) : \cdots : f_r(x_0, \ldots, x_r)\big).$$

When $\mathcal{F}$ is generically finite and I satisfies the property F_0, we have

$$\deg(\mathcal{F}) \leqslant \mu_1 \mu_2 \cdots \mu_r.$$

In addition, if I satisfies the condition G_{r+1} then

$$\deg(\mathcal{F}) = \mu_1 \mu_2 \cdots \mu_r.$$

Proof. Let the $\alpha_{i,j,k}$'s introduced in Notation 5.49 stand for the coefficients of the polynomials in the entries of the presentation matrix φ. Then, under Notation 5.49, there is a canonical isomorphism

$$\Phi : (A/\mathfrak{n})[x_0, \ldots, x_r] \xrightarrow{\cong} D = \mathbb{k}[x_0, \ldots, x_r]$$

which, when applied to the entries of the matrix M, yields the respective entries of the matrix φ. Thus it is equivalent to consider the rational map $\mathbb{g} : \mathbb{P}^r_{\mathbb{k}} \dashrightarrow \mathbb{P}^r_{\mathbb{k}}$ determined by the representative $(\overline{g_0} : \cdots : \overline{g_r})$ where $\Phi(\overline{g_i}) = f_i$.

Since $I_r(\mathbb{M})$ satisfies the condition G_{r+1} (Lemma 5.50), then Corollary 4.13 gives us that $\deg(\mathbb{G}) = \mu_1 \mu_2 \cdots \mu_r$.

Since $\mathcal{G}$ is generically finite by Lemma 5.50(ii) and Remark 5.24, its image is the whole of $\mathbb{P}^r_A$, the latter obviously being a normal scheme. In addition, since I satisfies F_0, the conditions of Theorem 5.44$(i)(iv)$ are satisfied, hence

$$\deg(\mathcal{F}) = \deg(\mathbb{g}) \leqslant \deg(\mathcal{G}) = \deg(\mathbb{G}) = \mu_1 \mu_2 \cdots \mu_r.$$

When I satisfies G_{r+1}, then the equality $\deg(\mathcal{F}) = \mu_1 \mu_2 \cdots \mu_r$ follows directly from Corollary 4.13. ☐

A particular satisfying case is when $\mathcal{F}$ is a plane rational map. In this case F_0 is not a constraint at all, and we recover the result of Proposition 3.47.

Corollary 5.52. *Let $\mathcal{F} : \mathbb{P}^2_{\mathbb{k}} \dashrightarrow \mathbb{P}^2_{\mathbb{k}}$ be a rational map defined by a perfect base ideal I of height* *two. Then,*

$$\deg(\mathcal{F}) \leqslant \mu_1\mu_2$$

and an equality is attained if I *is locally a complete intersection at its minimal primes.*

Proof. In this case property F_0 comes for free because $\mathrm{ht}(I_1(\varphi)) \geqslant \mathrm{ht}(I_2(\varphi)) = 2$ is always the case. Also, here l.c.i. at its minimal primes is equivalent to G_3. ☐

Finally, we show a simple family of plane rational maps where the degree of the map decreases arbitrarily under specialization. Also, in the following example, under general specialization of the coefficients the degree remains constant.

Example 5.53. *Let* $m \geqslant 1$ *be an integer. Let* $A = \Bbbk[a]$ *be a polynomial ring over* $\Bbbk$*. Let* $R = A[x,y,z]$ *be a standard graded polynomial ring over* A *and consider the following homogeneous matrix*

$$\mathcal{M} = \begin{pmatrix} x & zy^{m-1} \\ -y & zx^{m-1} + y^m \\ az & zx^{m-1} \end{pmatrix}$$

with entries in R*. For any* $\beta \in \Bbbk$*, let* $\mathfrak{n}_\beta := (a - \beta) \subset A$*. Let* $\mathcal{I} = (g_0, g_1, g_2) := I_2(\mathcal{M}) \subset R$ *and* $I_\beta := \left(\pi_\beta(g_0), \pi_\beta(g_1), \pi_\beta(g_2)\right) \subset R/\mathfrak{n}_\beta R$ *be the specialization of* $\mathcal{I}$ *via the canonical map* $\pi_\beta : R \twoheadrightarrow R/\mathfrak{n}_\beta R$.

Let $\mathcal{G} : \mathbb{P}^2_A \dashrightarrow \mathbb{P}^2_A$ *and* $\mathfrak{g}_\beta : \mathbb{P}^2_{A/\mathfrak{n}_\beta} \dashrightarrow \mathbb{P}^2_{A/\mathfrak{n}_\beta}$ *be rational maps with representatives* $(g_0 : g_1 : g_2)$ *and* $\left(\pi_\beta(g_0) : \pi_\beta(g_1) : \pi_\beta(g_2)\right)$*, respectively.*

When $\beta = 0$*, from* [67, Proposition 2.3] *we have that* $\mathfrak{g}_0 : \mathbb{P}^2_{A/\mathfrak{n}_0} \dashrightarrow \mathbb{P}^2_{A/\mathfrak{n}_0}$ *is a de Jonquières map, which is birational (also, see Theorem 3.59). On the other hand, if* $\beta \neq 0$*, then* I_β *satisfies the condition* G_3 *and so we have* $\deg\left(\mathfrak{g}_\beta\right) = m$.

Therefore, it follows that

$$\deg\left(\mathfrak{g}_\beta\right) = \begin{cases} 1 & \text{if } \beta = 0, \\ m & \text{if } \beta \neq 0. \end{cases}$$

Also, note that $\deg(\mathcal{G}) = m$.

Part III

Combinatorics

Chapter 6

Regularity and Gröbner bases of the Rees algebra of edge ideals of bipartite graphs

Let $G = (V(G), E(G))$ be a bipartite graph on the vertex set $V(G) = X \cup Y$ with bipartition $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Let $\mathbb{k}$ be a field and let R be the polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. The edge ideal $I = I(G)$, associated to G, is the ideal of R generated by the set of monomials $x_i y_j$ such that x_i is adjacent to y_j.

Let $\mathcal{R}(I) = \bigoplus_{i=0}^{\infty} I^i t^i \subset R[t]$ be the Rees algebra of the edge ideal I. Let $f_1, \ldots, f_q$ be the square free monomials of degree two generating I. We can see $\mathcal{R}(I)$ as a quotient of the polynomial ring $S = R[T_1, \ldots, T_q]$ via the map

$$\begin{gathered} S = \mathbb{k}[x_1, \ldots, x_n, y_1 \ldots, y_m, T_1, \ldots, T_q] \xrightarrow{\psi} \mathcal{R}(I) \subset R[t], \\ \psi(x_i) = x_i, \quad \psi(y_i) = y_i, \quad \psi(T_i) = f_i t. \end{gathered} \tag{6.1}$$

Then, the presentation of $\mathcal{R}(I)$ is given by $S/\mathcal{K}$ where $\mathcal{K} = \mathrm{Ker}(\psi)$. We give a bigraded structure to $S = \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_m] \otimes_{\mathbb{k}} \mathbb{k}[T_1, \ldots, T_q]$, where $\mathrm{bideg}(x_i) = \mathrm{bideg}(y_i) = (1, 0)$ and $\mathrm{bideg}(T_i) = (0, 1)$. The map ψ from (6.1) becomes bihomogeneous when we declare $\mathrm{bideg}(t) = (-2, 1)$. Then, we have that $S/\mathcal{K}$ and $\mathcal{K}$ have natural bigraded structures as S-modules.

The universal Gröbner basis of the ideal $\mathcal{K}$ is defined as the union of all the reduced Gröbner bases $\mathcal{G}_<$ of the ideal $\mathcal{K}$ as $<$ runs over all possible monomial orders (see [138]). In our first main result we compute the universal Gröbner basis of the defining equations $\mathcal{K}$ of the Rees algebra $\mathcal{R}(I)$.

From [147, Theorem 3.1, Proposition 3.1] we have a precise description of $\mathcal{K}$ given by the syzygies of I and the set even of closed walks in the graph G. The algebra $\mathcal{R}(I)$, as a bigraded S-module, has a minimal bigraded free resolution

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathcal{R}(I) \longrightarrow 0, \tag{6.2}$$

where $F_i = \oplus_j S(-a_{ij}, -b_{ij})$. In the same way as in [122], we can define the xy-regularity of $\mathcal{R}(I)$ by the integer

$$\operatorname{reg}_{xy}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} - i\},$$

or equivalently by

$$\operatorname{reg}_{xy}(\mathcal{R}(I)) = \max\{a \in \mathbb{Z} \mid \beta^S_{i,(a+i,b)}(\mathcal{R}(I)) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},$$

where $\beta^S_{i,(a,b)}(\mathcal{R}(I)) = \dim_{\Bbbk}(\operatorname{Tor}^S_i(\mathcal{R}(I), \Bbbk)_{(a,b)})$.

Similarly, we can define the T-regularity by

$$\operatorname{reg}_T(\mathcal{R}(I)) = \max_{i,j}\{b_{ij} - i\}$$

and the total regularity by

$$\operatorname{reg}(\mathcal{R}(I)) = \max_{i,j}\{a_{ij} + b_{ij} - i\}.$$

The aim of this chapter is to investigate different aspects of the Rees algebra $\mathcal{R}(I)$ of I. We compute its total regularity as a bigraded algebra and the universal Gröbner basis of its defining equations; interestingly, both of them are described in terms of the combinatorics of G. We apply these ideas to study the regularity of the powers of I. For any $s \geqslant \operatorname{match}(G) + |E(G)| + 1$ we prove that $\operatorname{reg}(I^{s+1}) = \operatorname{reg}(I^s) + 2$, and for all $s \geqslant 1$ we show that $\operatorname{reg}(I^s) \leqslant 2s + \operatorname{match}(G) - 1$.

6.1 The universal Gröbner basis of $\mathcal{K}$

In this section we give an explicit description of the universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$. Our approach is the following: first we compute the set of circuits of the incidence matrix of the cone graph, and then we translate this set of circuits into a description of $\mathcal{U}$.

The following will be assumed in most of this chapter.

Setup 6.1. *Let G be a bipartite graph with bipartition $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, and R be the polynomial ring $R = \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Let I be the edge ideal $I(G) = (f_1, \ldots, f_q)$ of G. We consider the Rees algebra $\mathcal{R}(I)$ as a quotient of $S = R[T_1, \ldots, T_q]$ by using (6.1). Let $\mathcal{K}$ be the defining equations of the Rees algebra $\mathcal{R}(I)$.*

Let $A = (a_{i,j}) \in \mathbb{R}^{n+m,q}$ be the incidence matrix of the graph G. Then we construct the matrix M of the following form

$$M = \begin{pmatrix} a_{1,1} & \ldots & a_{1,q} & e_1 & \ldots & e_{n+m} \\ \vdots & \vdots & \vdots & & & \\ a_{n+m,1} & \ldots & a_{n+m,q} & & & \\ 1 & \ldots & 1 & & & \end{pmatrix}, \tag{6.3}$$

where $e_1, \ldots e_{n+m}$ are the first $n+m$ unit vectors in $\mathbb{R}^{n+m+1}$ (see [56, Section 3] for more details). This matrix corresponds to the presentation of $\mathcal{R}(I)$ given in (6.1). For any vector $\beta \in \mathbb{Z}^{n+m+q}$ with nonnegative coordinates we shall use the notation

$$\mathbf{xyT}^{\beta} = x_1^{\beta_{q+1}} \cdots x_n^{\beta_{q+n}} y_1^{\beta_{q+n+1}} \cdots y_m^{\beta_{q+n+m}} T_1^{\beta_1} \cdots T_q^{\beta_q}.$$

A given vector $\alpha \in \mathrm{Ker}(M) \cap \mathbb{Z}^{n+m+q}$, can be written as $\alpha = \alpha^+ - \alpha^-$ where α^+ and α^- are nonnegative and have disjoint support.

Definition 6.2 ([138]). *A vector $\alpha \in \mathrm{Ker}(M) \cap \mathbb{Z}^{n+m+q}$ is called a circuit if it has minimal support supp(α) with respect to inclusion and its coordinates are relatively prime.*

Notation 6.3. *Given a walk $w = \{v_0, \ldots, v_a\}$, each edge $\{v_{j-1}, v_j\}$ corresponds to a variable T_{i_j}, and we set $T_{w^+} = \prod_{j \text{ is even}} T_{i_j}$ and $T_{w^-} = \prod_{j \text{ is odd}} T_{i_j}$ (in case $a = 1$ we make $T_{w^+} = 1$). We adopt the following notations:*

(i) Let $w = \{v_0, \ldots, v_a = v_0\}$ be an even cycle in G. Then by T_w we will denote the binomial $T_{w^+} - T_{w^-} \in \mathcal{K}$.

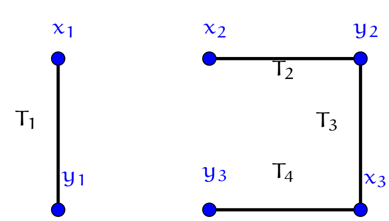
(ii) Let $w = \{v_0, \ldots, v_a\}$ be an even path in G, since G is bipartite then both endpoints of w belong to the same side of the bipartition, i.e. either $v_0 = x_i, v_a = x_j$ or $v_0 = y_i, v_a = y_j$. Then the path w determines the binomial

$$v_0 T_{w^+} - v_a T_{w^-} \in \mathcal{K}.$$

(iii) Let $w_1 = \{u_0, \ldots, u_a\}$, $w_2 = \{v_0, \ldots, v_b\}$ be two disjoint odd paths, then the endpoints of w_1 and w_2 belong to different sides of the bipartition. Let $T_{(w_1,w_2)^+} = T_{w_1^+} T_{w_2^-}$ and $T_{(w_1,w_2)^-} = T_{w_1^-} T_{w_2^+}$, then w_1 and w_2 determine the binomial

$$u_0 u_a T_{(w_1,w_2)^+} - v_0 v_b T_{(w_1,w_2)^-} \in \mathcal{K}.$$

Example 6.4. *In the bipartite graph shown below*



we have that the odd paths $w_1 = (x_1, y_1)$ and $w_2 = (x_2, y_2, x_3, y_3)$ determine the binomial $x_1 y_1 T_2 T_4 - x_2 y_3 T_1 T_3$.

Let $\mathcal{U}$ be the universal Göbner basis of $\mathcal{K}$. In general we have that the set of circuits is contained in $\mathcal{U}$ ([138, Proposition 4.11]). But from the fact that M is totally unimodular ([56, Theorem 3.1]), we can use [138, Proposition 8.11] and obtain the equality

$$\mathcal{U} = \{\mathbf{xyT}^{\alpha^+} - \mathbf{xyT}^{\alpha^-} \mid \alpha \text{ is a circuit of } M\}.$$

Therefore we shall focus on determining the circuits of M, and for this we will need to introduce the concept of the cone graph $C(G)$. The vertex set of the graph $C(G)$ is obtained by adding a new vertex z to G, and its edge set consists of the edges in $E(G)$ together with the edges $\{x_1, z\}, \ldots, \{x_n, z\}, \{y_1, z\}, \ldots, \{y_m, z\}$.

Theorem 6.5. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. The universal Gröbner basis $\mathcal{U}$ of $\mathcal{K}$ is given by*

$$\begin{aligned}\mathcal{U} = &\{T_w \mid w \textit{ is an even cycle}\}\\ &\cup \{v_0 T_{w^+} - v_a T_{w^-} \mid w = (v_0, \ldots, v_a) \textit{ is an even path}\}\\ &\cup \{u_0 u_a T_{(w_1,w_2)^+} - v_0 v_b T_{(w_1,w_2)^-} \mid w_1 = (u_0, \ldots, u_a) \textit{ and}\\ &\qquad\qquad\qquad w_2 = (v_0, \ldots, v_b) \textit{ are disjoint odd paths}\}.\end{aligned}$$

Proof. Let $\Bbbk[C(G)]$ be the monomial subring of the graph $C(G)$, which is generated by the monomials

$$\Bbbk[C(G)] = \Bbbk\big[\{x_i y_j \mid \{x_i, y_j\} \in E(G)\} \cup \{x_i z \mid i = 1, \ldots, n\} \cup \{y_i z \mid i = 1, \ldots, m\}\big].$$

As we did for the Rees algebra $\mathcal{R}(I)$, we can define a similar surjective homomorphism

$$\begin{gathered}\pi : S \longrightarrow \Bbbk[C(G)] \subset R[z],\\ \pi(x_i) = x_i z, \quad \pi(y_i) = y_i z, \quad \pi(T_i) = f_i.\end{gathered}$$

We have a natural isomorphism between $\mathcal{R}(I)$ and $\Bbbk[C(G)]$ [145, Excercise 7.3.3]. For instance, we can define the homomorphism $\varphi : R[t] \to R[z, z^{-1}]$ given by $\varphi(x_i) = x_i z$, $\varphi(y_i) = y_i z$ and $\varphi(t) = 1/z^2$, then the restriction $\varphi\mid_{\mathcal{R}(I)}$ of φ to $\mathcal{R}(I)$ will give us the required isomorphism because both algebras are integral domains of the same dimension (see Proposition 6.21 (i)).

Hence we will identify the ideal $\mathcal{K}$ with the kernel of π. Let N be the incidence matrix of the cone graph $C(G)$. From [147, Proposition 4.2], we have that a vector $\alpha \in \mathrm{Ker}(N) \cap \mathbb{Z}^{m+n+q}$ is a circuit of N if and only if the monomial walk defined by α corresponds to an even cycle or to two edge disjoint odd cycles joined by a path.

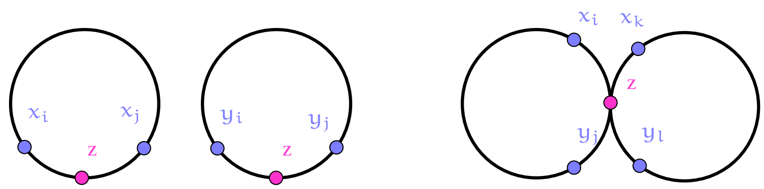
Since the graph G is bipartite, then an odd cycle in $C(G)$ will necessarily contain the vertex z. Therefore the monomial walks defined by the circuits of N are of the following types:

(i) An even cycle in $C(G)$ that does not contain the vertex z.

(ii) An even cycle in $C(G)$ that contains the vertex z.

(iii) Two odd cycles in $C(G)$ whose intersection is exactly the vertex z.

The figure below shows how the cases (ii) and (iii) may look.



(a) The two possible cycles of (ii). (b) The graph of (iii).

Since the circuits of the matrices M and N coincide, now we translate these monomial walks in $C(G)$ into binomials of $\mathcal{K}$. An even cycle in $C(G)$ not containing z, is also an even cycle in G, and it determines a binomial in $\mathcal{K}$ using Notation 6.3. In the cases (ii) and (iii), we delete vertex z in order to get a subgraph H of G. Thus we have that H is either an even path or two disjoint odd paths, and we translate these into binomials in $\mathcal{K}$ using Notation 6.3. $\square$

Remark 6.6. *Alternatively in Theorem 6.5, we can see that the matrices* M *and* N *have the same kernel because they are equivalent. We multiply the last row of* M *by* -2 *and then we successively add the rows* $1, \dots, n+m$ *to the last row; with these elementary row operations we transform* M *into* N.

Example 6.7. *Using Theorem 6.5, the universal Gröbner basis of the defining equations of the Rees algebra of the graph in Example 6.4 is given by*

$$\{x_2y_2T_1 - x_1y_1T_2,\ x_2y_3T_1T_3 - x_1y_1T_2T_4,\ x_3T_2 - x_2T_3,\ x_3y_2T_1 - x_1y_1T_3,\\ x_3y_3T_1 - x_1y_1T_4,\ y_3T_3 - y_2T_4,\ x_3y_3T_2 - x_2y_2T_4\}.$$

It can also be checked in [60] *using the command* `universalGroebnerBasis`.

Corollary 6.8. *Let* G *be a bipartite graph and* $I = I(G)$ *be its edge ideal. The universal Gröbner basis* $\mathcal{U}$ *of* $\mathcal{K}$ *consists of square free binomials with degree at most linear in the variables* x_i*'s and at most linear in the variables* y_i*'s.*

6.2 Upper bound for the xy-regularity

In this section we get an upper bound for the xy-regularity of $\mathcal{R}(I)$, and the important point is that we will choose a special monomial order. Using the xy-regularity we can find an upper bound for the regularity of all the powers of the edge ideal I.

Since most of the upper bounds for the regularity of the powers of edge ideals are based on the technique of even-connection [8], then a strong motivation for this section is trying to give new tools for the challenging conjecture:

Conjecture 6.9 (Alilooee, Banerjee, Beyarslan and Hà). *Let* G *be an arbitrary graph then*

$$\operatorname{reg}(I(G)^s) \leqslant 2s + \operatorname{reg}(I(G)) - 2$$

for all $s \geqslant 1$.

The following theorem will be crucial in our treatment.

Theorem 6.10. ([122, Theorem 5.3], [27, Theorem 3.5], [73, Proposition 10.1.6]) *The regularity of each power* I^s *is bounded by*

$$\operatorname{reg}(I^s) \leqslant 2s + \operatorname{reg}_{xy}(\mathcal{R}(I)).$$

By fixing a particular monomial order $<$ in S, then we can see the initial ideal $\operatorname{in}_<(\mathcal{K})$ as the special fibre of a flat family whose general fibre is $\mathcal{K}$ (see e.g. [73, Chapter 3] or [47, Chapter 15]), and we can get a bigraded version of [73, Theorem 3.3.4, (c)].

Theorem 6.11. *Let* $<$ *be a monomial order in* S*, then we have*

$$\operatorname{reg}_{xy}(\mathcal{R}(I)) \leqslant \operatorname{reg}_{xy}(S/\operatorname{in}_<(\mathcal{K})).$$

Let $\mathcal{M}$ be an arbitrary maximal matching in G with $|\mathcal{M}| = r$. We assume that the vertices of G are numbered in such a way that $\mathcal{M}$ consists of the edges

$$\mathcal{M} = \big\{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_r, y_r\}\big\},$$

and also we assume that $n = |X| \leqslant |Y| = m$.

In $R = \Bbbk[x_1 \ldots, x_n, y_1, \ldots, y_m]$ we consider the lexicographic monomial order induced by

$$x_n > \ldots > x_2 > x_1 > y_m > \ldots > y_2 > y_1.$$

We choose an arbitrary monomial order $<^\#$ on $\Bbbk[T_1, \ldots, T_q]$, then we define the following monomial order $<^{\mathcal{M}}$ on $S = \Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_m, T_1, \ldots, T_q]$: for two monomials $\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1}\mathbf{T}^{\gamma_1}$ and $\mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}\mathbf{T}^{\gamma_2}$ we have

$$\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1}\mathbf{T}^{\gamma_1} <^{\mathcal{M}} \mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}\mathbf{T}^{\gamma_2}$$

if either

(i) $\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1} < \mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}$ or

(ii) $\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1} = \mathbf{x}^{\alpha_2}\mathbf{y}^{\beta_2}$ and $\mathbf{T}^{\gamma_1} <^\# \mathbf{T}^{\gamma_2}$.

Let $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$ be the reduced Gröbner basis of $\mathcal{K}$ with respect to $<^{\mathcal{M}}$. The possible type of binomials inside $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$ were described in Theorem 6.5, now we focus on obtaining a more refined information about the type (iii) in Notation 6.3.

Notation 6.12. *In this section, for notational purposes (and without loss of generality) we shall assume that w_1 and w_2 are disjoint odd paths of the form*

$$w_1 = (x_e, u_1, \ldots, u_{2a}, y_f),$$
$$w_2 = (x_g, v_1, \ldots, v_{2b}, y_h).$$

Then we analyze the binomial $x_e y_f T_{(w_1,w_2)^+} - x_g y_h T_{(w_1,w_2)^-}$.

Lemma 6.13. *Let* $x_e y_f T_{(w_1,w_2)^+} - x_g y_h T_{(w_1,w_2)^-} \in \mathcal{G}_{<_\mathcal{M}}(\mathcal{K})$*, then we have*

(i) at least one of the vertices x_e, y_f *is in the matching* $\mathcal{M}$*, i.e.* $e \leqslant r$ *or* $f \leqslant r$*;*

(ii) at least one of the vertices x_g, y_h *is in the matching* $\mathcal{M}$*, i.e.* $g \leqslant r$ *or* $h \leqslant r$.

Proof. (i) First, assume that $a = 0$, i.e. w_1 has length one. Since $\mathcal{M}$ is a maximal matching then we necessarily get that $e \leqslant r$ or $f \leqslant r$.

Now let $a > 0$, and by contradiction assume that $e > r$ and $f > r$. From the maximality of $\mathcal{M}$, we get that $u_1 = y_j$ where $j \leqslant r$. We consider the even path

$$w_3 = (y_j, \ldots, u_{2a}, y_f),$$

then using Notation 6.3 we get the binomial

$$F = y_j T_{w_3^+} - y_f T_{w_3^-} \in \mathcal{K}.$$

We have $\text{in}_{<_\mathcal{M}}(F) = y_f T_{w_3^-}$ because $f > j$. So we obtain that $\text{in}_{<_\mathcal{M}}(F)$ divides $x_e y_f T_{(w_1,w_2)^+}$, and this contradicts that $\mathcal{G}_{<_\mathcal{M}}(\mathcal{K})$ is reduced.

(ii) Follows identically. $\square$

In the rest of this chapter we assume the following.

Notation 6.14. $b(G)$ *represents the minimum cardinality of the maximal matchings of* G *and* $\text{match}(G)$ *denotes the maximum cardinality of the matchings of* G

Theorem 6.15. *Let* G *be a bipartite graph and* $I = I(G)$ *be its edge ideal. The* xy*-regularity of* $\mathcal{R}(I)$ *is bounded by*

$$\text{reg}_{xy}(\mathcal{R}(I)) \leqslant \min\left\{|X| - 1,\ |Y| - 1,\ 2b(G) - 1\right\}.$$

Proof. From Theorem 6.11, it is enough to prove that

$$\text{reg}_{xy}(S/\text{in}_{<_\mathcal{M}}(\mathcal{K})) \leqslant \min\left\{|X| - 1,\ |Y| - 1,\ 2r - 1\right\}.$$

Let $\{m_1, \ldots, m_c\}$ be the monomials obtained as the initial terms of the elements of $\mathcal{G}_{<_\mathcal{M}}(\mathcal{K})$. We consider the Taylor resolution (see e.g. [73, Section 7.1])

$$0 \longrightarrow T_c \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow S/\mathrm{in}_{<_\mathcal{M}}(\mathcal{K}) \longrightarrow 0,$$

where each T_i as a bigraded S-module has the structure

$$T_i = \bigoplus_{1 \leqslant j_1 < \ldots < j_i \leqslant c} S\big(-\deg_{xy}(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i})), -\deg_T(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i}))\big).$$

From it, we get the upper bound

$$\mathrm{reg}_{xy}(S/\mathrm{in}_{<_\mathcal{M}}(\mathcal{K})) \leqslant \max\big\{ \deg_{xy}(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i})) - i \mid \{j_1, \ldots, j_i\} \subset \{1, \ldots, c\}\big\}.$$

When $\deg_{xy}(m_{j_i}) \leqslant 1$, then we have

$$\deg_{xy}(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i})) - i \leqslant \deg_{xy}(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_{i-1}})) - (i-1). \tag{6.4}$$

So, according with Theorem 6.5, we only need to consider subsets $\{j_1, \ldots, j_i\}$ such that for each $1 \leqslant k \leqslant i$ we have $m_{j_k} = \mathrm{in}_{<_\mathcal{M}}(F_k)$ and F_k is a binomial as in Notation 6.12. We use the notation $\mathrm{in}_{<_\mathcal{M}}(F_k) = x_{e_k} y_{f_k} B_k$, where B_k is a monomial in the T_i's. Also, we can assume that $x_{e_1}y_{f_1}, x_{e_2}y_{f_2}, \ldots, x_{e_k}y_{f_k}$ are pairwise relatively prime, because we can make a reduction like in (6.4) if this condition is not satisfied.

Thus, in order to finish the proof, we only need to show that we necessarily have

$$i \leqslant \min\{|X| - 1,\ |Y| - 1,\ 2r - 1\}$$

under the two previous conditions. Since the two paths that define each F_k are disjoint, then by the monomial order chosen we have that $e_k > 1$ for each k, and by a "pigeonhole" argument follows that $i \leqslant |X| - 1 \leqslant |Y| - 1$. Also, from Lemma 6.13 there are at most $2r - 1$ available positions to satisfy the condition of being co-primes. Thus we have $i \leqslant 2r - 1$, and the result of the theorem follows because $\mathcal{M}$ is an arbitrary maximal matching. $\square$

Corollary 6.16. *Let G be a bipartite graph and $I = I(G)$ be its edge ideal. For all $s \geqslant 1$ we have*

$$\mathrm{reg}(I^s) \leqslant 2s + \min\big\{|X| - 1,\ |Y| - 1,\ 2b(G) - 1\big\}.$$

Proof. It follows from Theorem 6.15 and Theorem 6.10. $\square$

Remark 6.17. *From the fact that* $\text{co-chord}(G) \leqslant \mathrm{match}(G) \leqslant \min\{|X|, |Y|\}$ *(see* [90]*) and* $\mathrm{match}(G) \leqslant 2b(G)$ *(see* [75, Proposition 2.1]*), then we have the following relations*

$$\text{co-chord}(G) - 1 \leqslant \mathrm{match}(G) - 1 \leqslant \min\big\{|X| - 1,\ |Y| - 1,\ 2b(G) - 1\big\}.$$

Although the last upper bound is weaker, it is interesting that an approach based on Gröbner bases can give a sharp answer in several cases.

In the last part of this section we deal with the case of a complete bipartite graph. The Rees algebra of these graphs was studied in [148].

Setup 6.18. *By* G *we will denote a complete bipartite graph with bipartition* $X = \{x_1, \ldots, x_n\}$ *and* $Y = \{y_1, \ldots, y_m\}$*. Let* $I = \{x_i y_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ *be the edge ideal of* G *and let* T_{ij} *be the variable that corresponds to the edge* $x_i y_j$*. Thus, we have a canonical map*

$$\begin{gathered} S = \mathbb{k}[x_i\text{'s},\ y_j\text{'s},\ T_{ij}\text{'s}] \xrightarrow{\psi} \mathcal{R}(I) \subset R[t], \\ \psi(x_i) = x_i, \quad \psi(y_i) = y_i, \quad \psi(T_{ij}) = x_i y_j t. \end{gathered} \tag{6.5}$$

Let $\mathcal{K}$ *be the kernel of this map. For simplicity of notation we keep the same monomial order* $<^{\mathcal{M}}$.

Exploiting our characterization of the universal Gröbner basis of $\mathcal{K}$, we shall prove that all the powers of the edge ideal of G have a linear free resolution.

Lemma 6.19. *Let* G *be a complete bipartite graph. The reduced Gröbner basis* $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$ *consists of binomials with linear* xy*-degree.*

Proof. From Theorem 6.5 we only need to show that any binomial determined by two disjoint odd paths is not contained in $\mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$. Let $x_e y_f T_{(w_1,w_2)^+} - x_g y_h T_{(w_1,w_2)^-}$ be a binomial like in Notation 6.12. By contradiction assume that $x_e y_f T_{(w_1,w_2)^+} - x_g y_h T_{(w_1,w_2)^-} \in \mathcal{G}_{<^{\mathcal{M}}}(\mathcal{K})$.

Without loss of generality we assume that $e > g$. Since G is complete bipartite, we choose the edge $x_e y_h$ and we append it to w_2, that is

$$w_3 = (x_g, v_1, \ldots, v_{2b}, y_h, x_e).$$

Using Notation 6.3 we get the binomial

$$F = x_g T_{w_3^+} - x_e T_{w_3^-} \in \mathcal{K},$$

with initial term $\text{in}_{<^{\mathcal{M}}}(F) = x_e T_{w_3^-}$ because $e > g$. Thus we get that $\text{in}_{<^{\mathcal{M}}}(F)$ divides $x_e y_f T_{(w_1,w_2)^+}$, a contradiction. ☐

Corollary 6.20. *Let* G *be a complete bipartite graph and* $I = I(G)$ *be its edge ideal. For all* $s \geqslant 1$ *we have* $\text{reg}(I^s) = 2s$.

Proof. Using Lemma 6.19 and repeating the same argument of Theorem 6.15 we can get $\text{reg}_{xy}(\mathcal{R}(I)) = 0$. Again, the result follows by Theorem 6.10. ☐

We remark that this previous result also follows from [90] since it is easy to check that co-chord$(G) = 1$ (i.e. it is a co-chordal graph) in the case of complete bipartite graphs.

6.3 The total regularity of $\mathcal{R}(I)$

In the previous sections we heavily exploited the fact that the matrix M (corresponding to $\mathcal{R}(I)$) is totally unimodular in the case of a bipartite graph G. From [56, Theorem 2.1] we have that $\mathcal{R}(I)$ is a normal domain, then a famous theorem by Hochster [76] (see e.g. [18, Theorem 6.10] or [19, Theorem 6.3.5]) implies that $\mathcal{R}(I)$ is Cohen-Macaulay. So, the Rees algebra $\mathcal{R}(I)$ of a bipartite graph G is also special from a more algebraic point of view (see [136]).

For notational purposes we let N be $N = n + m$. It is well known that the canonical module of S (with respect to our bigrading) is given by $S(-N, -q)$ (see e.g. [18, Proposition 6.26], or [19, Example 3.6.10] in the $\mathbb{Z}$-graded case). The Rees cone is the polyhedral cone of $\mathbb{R}^{N+1}$ generated by the set of vectors

$$\mathcal{A} = \{v \mid v \text{ is a column of } M \text{ in (6.3)}\},$$

and we will denote it by $\mathbb{R}_+\mathcal{A}$. The irreducible representation of the Rees cone for a bipartite graph was given in [56, Section 4].

Proposition 6.21. *Adopt Setup 6.1. The following statements hold:*

(i) The Krull dimension of $\mathcal{R}(I)$ *is* $\dim(\mathcal{R}(I)) = N + 1$.

(ii) The projective dimension of $\mathcal{R}(I)$ *as an* S*-module is equal to the number of edges minus one, that is,* $p = \operatorname{pd}_S(\mathcal{R}(I)) = q - 1$.

(iii) The canonical module of $\mathcal{R}(I)$ *is given by*

$$\omega_{\mathcal{R}(I)} = {}^*\mathrm{Ext}_S^p\big(\mathcal{R}(I),\ S(-N, -q)\big).$$

(iv) The bigraded Betti numbers of $\mathcal{R}(I)$ *and* $\omega_{\mathcal{R}(I)}$ *are related by*

$$\beta^S_{i,(a,b)}(\mathcal{R}(I)) = \beta^S_{p-i,(N-a,q-b)}(\omega_{\mathcal{R}(I)}).$$

Proof. (i) The Rees cone $\mathbb{R}_+\mathcal{A}$ has dimension $N + 1$ and the Krull dimension of $\mathcal{R}(I)$ is equal to this number (see e.g. [138, Lemma 4.2]). More generally, it also follows from Theorem 1.2.

Since clearly $\mathcal{R}(I)$ is a finitely generated S-module, then the statements (ii) and (iii) follow from [18, Theorem 6.28] (see [19, Proposition 3.6.12] for the $\mathbb{Z}$-graded case).

The statement (iv) follows from [18, Theorem 6.18]; also, see [18, page 224, equation 6.6]. □

Due to a formula of Danilov and Stanley (see e.g. [18, Theorem 6.31] or [19, Theorem 6.3.5]), the canonical module of $\mathcal{R}(I)$ is the ideal given by

$$\omega_{\mathcal{R}(I)} = \big(\{x_1^{a_1}\cdots x_n^{a_n} y_1^{a_{n+1}} \cdots y_N^{a_N} t^{a_{N+1}} \mid a = (a_i) \in (\mathbb{R}_+\mathcal{A})^\circ \cap \mathbb{Z}^{N+1}\}\big),$$

where $(\mathbb{R}_+\mathcal{A})^\circ$ denotes the topological interior of $\mathbb{R}_+\mathcal{A}$ with respect to the standard topology in R^{N+1}.

Now we can compute the total regularity of $\mathcal{R}(I)$.

Theorem 6.22. *Let* G *be a bipartite graph and* $I = I(G)$ *be its edge ideal. The total regularity of* $\mathcal{R}(I)$ *is given by*
$$\operatorname{reg}(\mathcal{R}(I)) = \operatorname{match}(G).$$

Proof. In the case of the total regularity, we can see $\mathcal{R}(I)$ as a standard graded S-module (i.e. $\deg(x_i) = \deg(y_i) = \deg(T_i) = 1$), and since $\mathcal{R}(I)$ is a Cohen-Macaulay S-module then the regularity can be computed with the last Betti numbers (see e.g. [126, page 283] or [47, Exercise 20.19]). Thus, from Proposition 6.21 we get

$$\begin{aligned}\operatorname{reg}(\mathcal{R}(I)) &= \max\left\{a + b - p \mid \beta^S_{p,(a,b)}(\mathcal{R}(I)) \neq 0\right\}\\ &= \max\left\{a + b - p \mid \beta^S_{0,(N-a,q-b)}(\omega_{\mathcal{R}(I)}) \neq 0\right\}\\ &= N + 1 - \min\left\{a + b \mid \beta^S_{0,(a,b)}(\omega_{\mathcal{R}(I)}) \neq 0\right\},\end{aligned}$$

and by the bigrading that we are using ($\operatorname{bideg}(x_i) = \operatorname{bideg}(y_i) = (1, 0)$ and $\operatorname{bideg}(t) = (-2, 1)$) then we obtain

$$\operatorname{reg}(\mathcal{R}(I)) = N + 1 - \min\left\{a_1 + \cdots + a_N - a_{N+1} \mid a = (a_i) \in (\mathbb{R}_+\mathcal{A})^\circ \cap \mathbb{Z}^{N+1}\right\}.$$

One can check that the number

$$-\min\left\{a_1 + \cdots + a_N - a_{N+1} \mid a = (a_i) \in (\mathbb{R}_+\mathcal{A})^\circ \cap \mathbb{Z}^{N+1}\right\}$$

coincides with the a-invariant of $\mathcal{R}(I)$ with respect to the $\mathbb{Z}$-grading induced by $\deg(x_i) = \deg(y_i) = 1$ and $\deg(t) = -1$. This last formula can be evaluated with the irreducible representation of the Rees cone [56, Corollary 4.3], it was done in [56, Proposition 4.5], and from it we get

$$\operatorname{reg}(\mathcal{R}(I)) = N - \beta_0,$$

where β_0 denotes the maximal size of an independent set of G. The minimal size of a vertex cover is equal to $N - \beta_0$, and we finally get

$$\operatorname{reg}(\mathcal{R}(I)) = \operatorname{match}(G)$$

from König's theorem. □

The following bound was obtained for the first power of the edge ideal in [63, Theorem 6.7].

Corollary 6.23. *Let* G *be a bipartite graph and* $I = I(G)$ *be its edge ideal. For all* $s \geqslant 1$ *we have*

$$\operatorname{reg}(I^s) \leqslant 2s + \operatorname{match}(G) - 1.$$

Proof. It is enough to prove that $\text{reg}_{xy}(\mathcal{R}(I)) \leqslant \text{reg}(\mathcal{R}(I)) - 1$. In the minimal bigraded free resolution (6.2) of $\mathcal{R}(I)$, suppose that $\text{reg}_{xy}(\mathcal{R}) = a_{ij} - i$ for some $i, j \in \mathbb{N}$. Since necessarily $b_{ij} \geqslant 1$ and

$$a_{ij} + b_{ij} - i \leqslant \text{reg}(\mathcal{R}(I)),$$

then we get the expected inequality. □

This previous upper bound is sharp in some cases (see [10, Lemma 4.4]). In the following corollary we get information about the eventual linearity.

Corollary 6.24. *Let* G *be a bipartite graph and* $I = I(G)$ *be its edge ideal. For all* $s \geqslant \text{match}(G) + q + 1$ *we have*

$$\text{reg}(I^{s+1}) = \text{reg}(I^s) + 2.$$

Proof. With the same argument of Corollary 6.23 we can prove that $\text{reg}_T(\mathcal{R}(I)) \leqslant \text{reg}(\mathcal{R}(I))$, here the difference is that in the minimal bigraded free resolution (6.2) we can have free modules of the type $S(0, -b_{ij})$ (for instance, in the syzygies of $\mathcal{R}(I)$ the ones that come from even cycles). Then the statement of the corollary follows from [42, Proposition 3.7]. □

6.4 Some final thoughts

In the last part of this chapter we give some ideas and digressions about Conjecture 6.9. Using a "refined Rees approach" with respect to the one of this chapter, we might get an answer to this conjecture for general graphs or perhaps for special families of graphs:

- Restricting the minimal bigraded free resolution (6.2) of $\mathcal{R}(I)$ to a graded T-part gives an exact sequence

  $$0 \longrightarrow (F_p)_{(*,k)} \longrightarrow \cdots \longrightarrow (F_1)_{(*,k)} \longrightarrow (F_0)_{(*,k)} \longrightarrow (\mathcal{R}(I))_{(*,k)} \longrightarrow 0$$

  for all k. This gives a (possibly non-minimal) graded free R-resolution of

  $$(\mathcal{R}(I))_{(*,k)} \cong I^k(2k).$$

  But in the case $k = 1$ we can check that

  $$0 \longrightarrow (F_p)_{(*,1)} \longrightarrow \cdots \longrightarrow (F_1)_{(*,1)} \longrightarrow (F_0)_{(*,1)} \longrightarrow I(2) \longrightarrow 0$$

  is indeed the minimal free resolution of $I(2)$. Thus, we can read the regularity I from (6.2), and a solution to Conjecture 6.9 can be given by proving that

  $$\max_{i,j} \{a_{ij} - i\} = \max_{i,j} \{a_{ij} - i \mid b_{ij} = 1\}.$$

- For bipartite graphs, Gröbner bases techniques can give very good results (for instance, in the case of complete bipartite graphs). Perhaps, for special families of bipartite graphs we can give “good” monomial orders.

- The existence of a canonical module in the case of bipartite graphs could give more information about the minimal bigraded free resolution of $\mathcal{R}(I)$. From [18, Theorem 7.26], we have that the maximal xy-degree and the maximal T-degree on each F_i of (6.2) form weakly increasing sequences of integers, that is

$$\max_j\{a_{ij}\} \leqslant \max_j\{a_{i+1,j}\} \quad \text{and} \quad \max_j\{b_{ij}\} \leqslant \max_j\{b_{i+1,j}\}$$

  (see e.g. [47, Exercise 20.19] for the $\mathbb{Z}$-graded case). Thus a more detailed analysis of the polyhedral geometry of the Rees cone $\mathbb{R}_+\mathcal{A}$ could give better results.

Chapter 7

Regularity of bicyclic graphs and their powers

In this chapter, we study the regularity of the edge ideal and its powers in the case of a bicyclic graph. Let $I(G)$ be the edge ideal of a bicyclic graph G with a dumbbell as the base graph. We characterize the Castelnuovo-Mumford regularity of $I(G)$ in terms of the induced matching number of G. For the base case of this family of graphs, i.e. dumbbell graphs, we explicitly compute the induced matching number and the regularity of the edge ideal $I(G)$. Moreover, we prove that $\operatorname{reg} I(G)^q = 2q + \operatorname{reg} I(G) - 2$, for all $q \geqslant 1$, when G is a dumbbell graph with a connecting path having no more than two vertices.

Note. The results of this chapter are based on joint work with Sepehr Jafari, Beatrice Picone and Navid Nemati.

7.1 Preliminaries

Let $R = \mathbb{k}[x_1, \dots, x_r]$ be a standard graded polynomial ring over a field $\mathbb{k}$ and let $\mathfrak{m} = (x_1, \dots, x_r)$ be its maximal irrelevant ideal. For a graded R-module M, one can define the Castelnuovo-Mumford regularity in different ways. We recall the definition of the regularity of an R-module M given in terms of the minimal free resolution of M. The *minimal graded free resolution of* M is an exact sequence of the form

$$0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to M \to 0,$$

where each F_i is a graded free R-module of the form $F_i = \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{i,j}(M)}$, each $\varphi_i : F_i \to F_{i-1}$, with $F_{-1} := M$, is a graded homomorphism of degree zero such that $\varphi_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ for all $i \geqslant 0$. The number $\beta_{i,j}(M)$, called the $(i,j)^{\text{th}}$-graded Betti number of M, is an important invariant of the module M. In particular, the number $\beta_i(M) = \sum_{j\in\mathbb{N}} \beta_{i,j}(M)$ is called the *i-th Betti number of* M. Note that the minimal free resolution of M is unique up to isomorphism, hence the graded Betti numbers are uniquely determined.

Definition 7.1. *Let M be a finitely generated graded R-module. The regularity of M is given by*

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Remark 7.2. *Note that, if I is a graded ideal of R, then* $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$.

Let $G = (V, E)$ be a graph with vertex set $V = \{x_1, \ldots, x_r\}$. Here, we recall some classes of graphs that we need for this study.

Definition 7.3. *Let $G = (V, E)$ be a graph.*

(i) G *is called a path on l vertices, denoted by P_l, if $V = \{x_1, \ldots, x_l\}$ and $\{x_i, x_{i+1}\} \in E$ for all* $1 \leqslant i \leqslant l - 1$.

(ii) G *is called a cycle on n vertices, denoted by C_n, if $V = \{x_1, \ldots, x_n\}$ and $\{x_i, x_{i+1}\} \in E$ for all $1 \leqslant i \leqslant n - 1$ and* $\{x_n, x_1\} \in E$.

(iii) G *is called a dumbbell graph if G contains two cycles C_n and C_m joined by a path P_l on l vertices. We denote it by $C_n \cdot P_l \cdot C_m$ (see Example 7.24).*

For a vertex u in a graph $G = (V, E)$, let $N_G(u) = \{v \in V | \{u, v\} \in E\}$ be the set of *neighbors* of u, and set $N_G[u] := N_G(u) \cup \{u\}$. An edge e is *incident* to a vertex u if $u \in e$. The degree of a vertex $u \in V$, denoted by $\deg_G(u)$, is the number of edges incident to u. When there is no confusion, we omit G and write $N(u), N[u]$ and $\deg(u)$. For an edge e in a graph $G = (V, E)$, we define $G \setminus e$ to be the subgraph of G obtained by deleting e from E (but the vertices remain in the graph).

Let $G = (V, E)$ be a graph and $W \subseteq V$, the *induced subgraph of* G *on* W, denoted by $G[W]$, is the graph with vertex set W and edge set $\{e \in E \mid e \subseteq W\}$. For a subset $W \subseteq V$ of the vertices in G, we define $G \setminus W$ to be the induced subgraph of G obtained by deleting the vertices of W and their incident edges from G. When $W = \{u\}$ consists of a single vertex, we write $G \setminus u$ instead of $G \setminus \{u\}$. For an edge $e = \{u, v\} \in E$, let $N_G[e] = N_G[u] \cup N_G[v]$ and define G_e to be the induced subgraph of G over the vertex set $V \setminus N_G[e]$.

One can think of the vertices of $G = (V, E)$ as the variables of the polynomial ring $R = \Bbbk[x_1, \ldots, x_r]$ for convenience. Similarly, the edges of G can be considered as square free monomials of degree two. By an abuse of notation, we use e to refer to both the edge $e = \{x_i, x_j\} \in E$ and the monomial $e = x_i x_j \in I(G)$.

Definition 7.4. *Let $G = (V, E)$ be a graph. A collection C of edges of G is called a matching if the edges in C are pairwise disjoint. The maximum size of a matching in G is called its matching number, which is denoted by* $\operatorname{match}(G)$.

A collection C *of edges of* G *is called an induced matching if* C *is a matching, and* C *consists of all edges of the induced subgraph* $G\big[\bigcup_{e\in C} e\big]$ *of* G*. The maximum size of an induced matching in* G *is called its induced matching number and it is denoted by* $\nu(G)$*.*

Remark 7.5. [10, Remark 2.12] *Let* P_l *be a path on* l *vertices, then we have*

$$\nu(P_l) = \left\lfloor \frac{l+1}{3} \right\rfloor$$

Remark 7.6. [10, Remark 2.13] *Let* C_n *be a cycle on* n *vertices, then we have*

$$\nu(C_n) = \left\lfloor \frac{n}{3} \right\rfloor.$$

Depending on $r = n \bmod 3$ *we can assume the following:*

(i) when $r = 0$*, there exists a maximal induced matching of* C_n *that does not contain the edges* x_1x_2 *and* x_1x_n*;*

(ii) when $r = 1$*, there exists a maximal induced matching of* C_n *that does not contain the edges* x_1x_2*,* x_1x_n *and* $x_{n-1}x_n$*;*

(iii) when $r = 2$*, there exists a maximal induced matching of* C_n *that does not contain the edges* x_1x_2*,* x_2x_3*,* x_1x_n *and* $x_{n-1}x_n$*.*

Theorem 7.7. [62, Lemma 3.1, Theorems 3.4 and 3.5] *Let* $G = (V, E)$ *be a graph.*

(i) If H *is an induced subgraph of* G*, then* $\operatorname{reg} I(H) \leqslant \operatorname{reg} I(G)$;

(ii) Let $x \in V$*, then*
$$\operatorname{reg} I(G) \leqslant \max\{\operatorname{reg} I(G \setminus x), \operatorname{reg} I(G \setminus N[x]) + 1\};$$

(iii) Let $e \in E$*, then*
$$\operatorname{reg} I(G) \leqslant \max\{2, \operatorname{reg} I(G \setminus e), \operatorname{reg} I(G_e) + 1\}.$$

Now we recall the concept of even-connection introduced by Banerjee in [8].

Definition 7.8 ([8])**.** *Let* $G = (V, E)$ *be a graph with edge ideal* $I = I(G)$*. Two vertices* x_i *and* x_j *in* G *are called even-connected with respect to an* s*-fold product* $M = e_1 \cdots e_s$*, where* $e_1, \ldots, e_s$ *are edges in* G*, if there is a path* $p_0, \ldots, p_{2l+1}$*, for some* $l \geqslant 1$*, in* G *such that the following conditions hold:*

(i) $p_0 = x_i$ *and* $p_{2l+1} = x_j$*;*

(ii) for all $0 \leqslant j \leqslant l-1, \{p_{2j+1}, p_{2j+2}\} = e_i$ *for some* i*;*

(iii) for all i*,* $\big|\{j \mid \{p_{2j+1}, p_{2j+2}\} = e_i\}\big| \leqslant \big|\{t \mid e_t = e_i\}\big|$*.*

Theorem 7.9. [8, Theorems 6.1 and 6.5] *Let* $M = e_1 e_2 \cdots e_s$ *be a minimal generator of* I^s. *Then* $(I^{s+1} : M)$ *is minimally generated by monomials of degree* 2*, and* uv *(*u *and* v *may be the same) is a minimal generator of* $(I^{s+1} : M)$ *if and only if either* $\{u, v\} \in E$ *or* u *and* v *are even-connected with respect to* M.

Remark 7.10. [8, Lemma 6.11] *Let* $(I^{s+1} : M)^{pol}$ *be the polarization of the ideal* $(I^{s+1} : M)$ *(see e.g.* [73, §1.6]*). From the previous theorem we can construct a graph* G' *whose edge ideal is given by* $(I^{s+1} : M)^{pol}$. *The new graph* G' *is given by:*

(i) All the vertices and edges of G.

(ii) Any two vertices u, v, $u \neq v$ *that are even-connected with respect to* M *are connected by an edge in* G'.

(iii) For every vertex u *which is even-connected to itself with respect to* M*, there is a new vertex* u' *which is connected to* u *by an edge and not connected to any other vertex (so* uu' *is a whisker).*

Theorem 7.11. [8, Theorem 5.2] *Let* G *be a graph and* $\{m_1, \ldots, m_r\}$ *be the set of minimal monomial generators of* $I(G)^q$ *for all* $q \geqslant 1$*, then*

$$\operatorname{reg} I(G)^{q+1} \leqslant \max\{\operatorname{reg}(I(G)^q : m_l) + 2q, 1 \leqslant l \leqslant r, \operatorname{reg} I(G)^q\}.$$

We recall a result on the regularity of monomial ideals.

Theorem 7.12. ([94], [72]) *Let* $I_1, \ldots, I_s$ *be monomial ideals in* R*, then*

$$\operatorname{reg}\left(R \Big/ \sum_{i=1}^{s} I_i\right) \leqslant \sum_{i=1}^{s} \operatorname{reg}(R/I_i).$$

In the particular case of edge ideals we have the following upper bound.

Corollary 7.13. *Let* G *be a simple graph. If* $G_1, \ldots, G_s$ *are subgraphs of* G *such that* $E(G) \subset \bigcup_{i=1}^{s} E(G_i)$ *then*

$$\operatorname{reg}(R/I(G)) \leqslant \sum_{i=1}^{s} \operatorname{reg}(R/I(G_i)).$$

The previous upper bound is sharp when G is a disjoint union of the graphs $G_1, \ldots, G_s$.

Corollary 7.14. [9, Corollary 3.10] *Let* G *be a simple graph. If* G *can be written as a disjoint union of graphs* $G_1, \ldots, G_s$ *then*

$$\operatorname{reg}(R/I(G)) = \sum_{i=1}^{s} \operatorname{reg}(R/I(G_i)).$$

The regularity of the edge ideal of a forest was first computed by Zheng in [152, Theorem 2.18].

Theorem 7.15. [152, Theorem 2.18] *Let* G *be a forest, then*

$$\operatorname{reg} I(G) = \nu(G) + 1.$$

In [96] Katzman proved that the previous equality is a lower bound for any graph.

Theorem 7.16. [96, Corollary 1.2] *Let* G *be a graph, then*

$$\operatorname{reg} I(G) \geqslant \nu(G) + 1.$$

The decycling number of a graph is an important combinatorial invariant which can be used to obtain an upper bound for the regularity of the edge ideal of a graph.

Definition 7.17. *For a graph* G *and* $D \subset V(G)$*, if* $G \setminus D$ *is acyclic, i.e. contains no induced cycle, then* D *is said to be a decycling set of* G*. The size of a smallest decycling set of* G *is called the decycling number of* G *and denoted by* $\nabla(G)$*.*

Theorem 7.18. [11, Theorem 4.11] *Let* G *be a graph, then*

$$\operatorname{reg} I(G) \leqslant \nu(G) + \nabla(G) + 1.$$

In [10] Beyarslan, Hà and Trung provided a formula for the regularity of the powers of edge ideals of forests and cycles in terms of the induced matching number.

Theorem 7.19. [10, Theorem 4.7] *Let* G *be a forest, then*

$$\operatorname{reg} I(G)^q = 2q + \nu(G) - 1.$$

for all $q \geqslant 1$.

Theorem 7.20. [10, Theorem 5.2]. *Let* C_n *be a cycle with* n *vertices, then*

$$\operatorname{reg} I(C_n) = \begin{cases} \nu(C_n) + 1 & \textit{if } n \equiv 0, 1 \pmod 3, \\ \nu(C_n) + 2 & \textit{if } n \equiv 2 \pmod 3, \end{cases}$$

where $\nu(C_n) = \lfloor \frac{n}{3} \rfloor$ *denote the induced matching number of* C_n*. Moreover,*

$$\operatorname{reg} I(C_n)^q = 2q + \nu(C_n) - 1.$$

and for all $q \geqslant 2$.

In addition, the authors of [10] gave a lower bound for the regularity of the powers of the edge ideal of an arbitrary graph, and an upper bound for the regularity of the edge ideal of a graph containing a Hamiltonian path.

Theorem 7.21. [10, Theorem 4.5] *Let G be a graph and let $\nu(G)$ denote its induced matching number. Then, for all $q \geqslant 1$, we have*

$$\operatorname{reg} I(G)^q \geqslant 2q + \nu(G) - 1.$$

Theorem 7.22. [10, Theorem 3.1] *Let G be a graph on n vertices. If G contains a Hamiltonian path, then*

$$\operatorname{reg} I(G) \leqslant \left\lfloor \frac{n+1}{3} \right\rfloor + 1.$$

Finally, we recall the Lozin transformation that will be an important tool in this chapter.

Definition 7.23 ([107])**.** *Let G be a graph and x be a vertex in G. We define a graph transformation as follows:*

(i) partition the neighborhood $N_G(x)$ of the vertex x into two subsets Y and Z in arbitrary way;

(ii) delete vertex x from the graph together with its incident edges;

(iii) add a path $P_4 = (y, a, b, z)$ to the rest of the graph;

(iv) connect the vertex y of the P_4 to each vertex in Y, and connect z to each vertex in Z.

We denote the transformed graph by $\mathcal{L}_x(G)$.

7.2 Regularity and induced matching number of a dumbbell graph

In this section we compute the induced matching number of a dumbbell graph and the regularity of its edge ideal. Recall that $C_n \cdot P_l \cdot C_m$ denotes the graph constructed by joining two cycles C_n and C_m via a path P_l. In this section, we denote the vertices of C_n, C_m and P_l by $\{x_1, \dots, x_n\}$, $\{y_1, \dots, y_m\}$ and $\{z_1, \dots, z_l\}$, respectively. We make the identifications $x_1 = z_1$ and $y_1 = z_l$.

Example 7.24. *Two base cases when $l = 2$ and $l = 1$ are the following:*

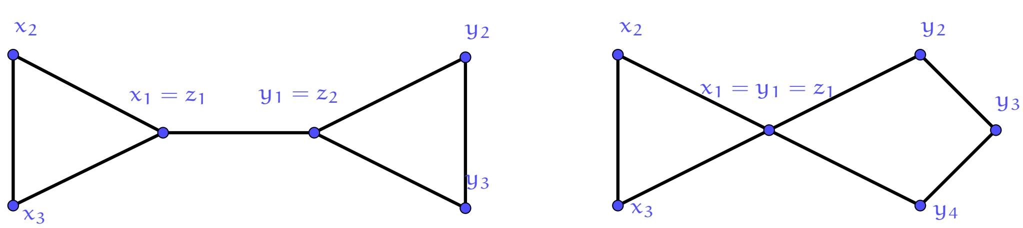
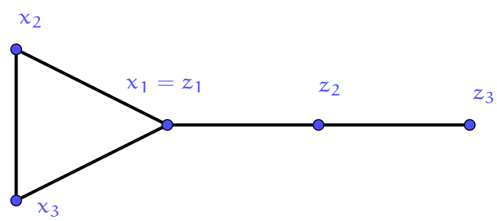


Figure 7.1: The graphs $C_3 \cdot P_2 \cdot C_3$ and $C_3 \cdot P_1 \cdot C_4$.

Notation 7.25. *Let ξ_3 be the function defined as below*

$$\xi_3(n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Let $C_n \cdot P_l$ be the graph given by connecting the path P_l to the cycle C_n. For instance, the graph $C_3 \cdot P_3$ can be illustrated as the following:



Proposition 7.26. *Let $n \geqslant 3$ and $l \geqslant 1$, then*

$$\nu(C_n \cdot P_l) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) + 1}{3} \right\rfloor.$$

Proof. Case 1: From Remark 7.6, in the case $n \equiv 2 \pmod 3$ we have that in clockwise and anticlockwise directions the two consecutive edges to the vertex x_1 are not chosen in a maximal induced matching of C_n. Then, we can choose the edges in P_l without any constraint coming from the maximal induced matching chosen in C_n, and so we have $\nu(C_n \cdot P_l) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{l+1}{3} \rfloor$.

Case 2: It remains to consider the case $\xi_3(n) = 1$, i.e., $n \equiv 0, 1 \pmod 3$. Let $\mathcal{M}$ be an induced matching of maximal size in G. We analyze separately the two cases of whether $z_1 z_2$ (the edge adjacent to the cycle C_n) is in $\mathcal{M}$ or not.

Suppose $z_1 z_2$ is not an edge of $\mathcal{M}$. Then $\mathcal{M}$ can be considered as the union of a maximal matching of C_n as introduced in Remark 7.6 and a maximal matching of the path $P_l \setminus z_1$. Thus $|\mathcal{M}| = \nu(C_n) + \nu(P_{l-1}) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor$.

If $z_1 z_2 \in \mathcal{M}$, then none of the edges incident to the vertices in $N_{C_n}[x_1] = \{x_1, x_2, x_n\}$ are in $\mathcal{M} |_{C_n} := \{e \in \mathcal{M} \mid e \subset C_n\}$. Hence $|\mathcal{M} |_{C_n}| = \nu(P_{n-3})$, and since $n \equiv 0, 1 \pmod 3$ then it follows $|\mathcal{M} |_{C_n}| = \lfloor \frac{n-2}{3} \rfloor = \lfloor \frac{n}{3} \rfloor - 1$. From $z_1 z_2 \in \mathcal{M}$ we have $|\mathcal{M} |_{P_l}| = \nu(P_l) = \lfloor \frac{l+1}{3} \rfloor$. So, by joining both computations we get $|\mathcal{M}| = \lfloor \frac{n}{3} \rfloor - 1 + \lfloor \frac{l+1}{3} \rfloor = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{l-2}{3} \rfloor$.

Therefore, we obtain that $\nu(C_n \cdot P_l) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{(l-1)+1}{3} \rfloor$. □

Theorem 7.27. *If $n, m \geqslant 3$ and $l \geqslant 1$, then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Proof. We use the same argument as in Proposition 7.26. By Remark 7.6 we have that when either $n \equiv 2 \pmod 3$ or $m \equiv 2 \pmod 3$, the maximal induced matching in C_n or in C_m does not affect the way we choose edges in the path P_l.

In the case $n \equiv 0, 1 \pmod 3$, we can choose a maximal induced matching that does not use the edge $z_1 z_2$ by Remark 7.6, i.e., the extreme vertex z_1 on the path P_l does not appear in the induced matching. Similarly, when $m \equiv 0, 1 \pmod 3$ we can drop the other extreme vertex. □

The aim of the rest of this section is to explicitly compute the regularity of $I(C_n \cdot P_l \cdot C_m)$ in terms of the induced matching number. We divide it into three subsections depending on the value of l mod 3. The base of our computations is given by the following proposition.

Proposition 7.28. *Let* $n, m \geqslant 3$ *and* $l \geqslant 1$*, then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) - \nu(C_n \cdot P_l \cdot C_m) = \operatorname{reg} I(C_n \cdot P_{l+3} \cdot C_m) - \nu(C_n \cdot P_{l+3} \cdot C_m).$$

Proof. From the formula obtained in Theorem 7.27 or [107, Lemma 1], we have the equality

$$\nu(C_n \cdot P_{l+3} \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1.$$

We can apply the Lozin transformation (see e.g. [107], [11]) to any of the vertices in the bridge P_l. Then, from [11, Theorem 1.1] we have

$$\operatorname{reg} I(C_n \cdot P_{l+3} \cdot C_m) = \operatorname{reg} I(C_n \cdot P_l \cdot C_m) + 1.$$

Thus, the statement of the proposition follows by subtracting these equalities. □

From the previous proposition, it follows that we only need to consider the cases $l = 1$, $l = 2$ and $l = 3$. We treat each case in a separate subsection.

The basic approach in the next three subsections is to obtain lower and upper bounds that coincide.

The case $l = 1$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_1 \cdot C_m$.

Proposition 7.29. *Let* $n, m \geqslant 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \leqslant \max\left\{ \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2 \right\}.$$

Moreover, $\operatorname{reg} I(C_n \cdot P_1 \cdot C_m)$ *is equal to one of these terms.*

Proof. We use [44, Lemma 3.2] that gives an improved version of the exact sequence that comes from deleting the vertex z_1 and its neighbors. We have

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) \in \Big\{ \operatorname{reg} I\big((C_n \cdot P_1 \cdot C_m) \setminus z_1\big), \operatorname{reg} I\big((C_n \cdot P_1 \cdot C_m) \setminus N[z_1]\big) + 1 \Big\}.$$

Since $(C_n \cdot P_1 \cdot C_m) \setminus z_1 = P_{n-1} \cup P_{m-1}$ and $(C_n \cdot P_1 \cdot C_m) \setminus N[z_1] = P_{n-3} \cup P_{m-3}$, we get the result by applying Theorem 7.15. $\square$

Theorem 7.30. *Let* $n, m \geqslant 3$*, then*

$$\operatorname{regI}(C_n \cdot P_1 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_1 \cdot C_m) + 2 & \text{if } n \equiv 2 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_1 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$. Since $\lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k}{3} \rfloor$ when $k \equiv 2 \pmod 3$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Thus, Proposition 7.29 yields

$$\operatorname{regI}(C_n \cdot P_1 \cdot C_m) \leqslant \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2. \tag{7.1}$$

Consider the induced subgraph $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\}$ where x_n is in C_n and it is incident to x_1 (e.g. see x_3 in Example 7.24). In fact, H is the graph given by joining C_m and a path P_{n-1}, that is, $H = C_m \cdot P_{n-1}$. Now, from Proposition 7.26, we have that $\nu(H) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. By Theorem 7.7(i), we get $\operatorname{regI}(C_n \cdot P_1 \cdot C_m) \geqslant \operatorname{regI}(H)$. From [4, Theorem 1.2], we have $\operatorname{regI}(H) = \nu(H) + 2$. Therefore, the equality holds in (7.1). The proof of this part is complete since Theorem 7.27 yields $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$.

For any case distinct to $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$, we have

$$\max\{\lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1, \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{m-2}{3} \rfloor + 2\} = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1.$$

Therefore, from Proposition 7.29, we have

$$\operatorname{regI}(C_n \cdot P_1 \cdot C_m) \leqslant \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1. \tag{7.2}$$

From Theorem 7.27, we have $\nu(C_n \cdot P_1 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor$. Moreover, Theorem 7.16 gives $\operatorname{regI}(C_n \cdot P_1 \cdot C_m) \geqslant \nu(C_n \cdot P_1 \cdot C_m) + 1$. Thus, the equality in (7.2) holds. So, the proof follows. $\square$

The case $l = 2$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_2 \cdot C_m$.

Remark 7.31. *From Theorem 7.20 we deduce that* $\operatorname{regI}(C_n) = \lfloor \frac{n-2}{3} \rfloor + 2$*. Similarly, we have* $\operatorname{reg}(R/I(C_n)) = \lfloor \frac{n-2}{3} \rfloor + 1$.

Proposition 7.32. *Let* $n, m \geqslant 3$*, then*

$$\nu(C_n \cdot P_2 \cdot C_m) \leqslant \operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) \leqslant \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2. \tag{7.3}$$

Proof. We only need to prove the inequality on the right since the lower bound is given due to Theorem 7.16. In the original graph $C_n \cdot P_2 \cdot C_m$ we remove the edge that connects the two cycles C_n and C_m. The set of vertices of C_n and C_m are given respectively by $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$, and we assume that the edge $e = x_1y_1$ is the bridge between the two cycles. We denote by $C_n \cup C_m$ the resulting graph given as the disjoint union of the two cycles C_n and C_m.

Note that $(C_n \cdot P_2 \cdot C_m) \setminus e = C_n \cup C_m$ and that $I\left((C_n \cdot P_2 \cdot C_m)_e\right) = I(P_{n-3} \cup P_{m-3})$ because $N_G[x_1] \cup N_G[y_1] = \{x_1, x_2, x_n, y_1, y_2, y_m\}$, where P_{n-3} is the path on the vertices $\{x_3, \ldots, x_{n-1}\}$ and P_{m-3} is the path on the vertices $\{y_3, \ldots, y_{m-1}\}$. Thus, Theorem 7.7(iii) gives the inequality

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) \leqslant \max\left\{\operatorname{reg}\left(\frac{R}{I(C_n \cup C_m)}\right), \operatorname{reg}\left(\frac{R}{I(P_{n-3} \cup P_{m-3})}\right) + 1\right\}.$$

From Corollary 7.14 and Remark 7.31, it follows that

$$\operatorname{reg}\left(\frac{R}{I(C_n \cup C_m)}\right) = \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 2.$$

Similarly, Theorem 7.15 and Remark 7.5 give that

$$\operatorname{reg}\left(\frac{R}{I(P_{n-3} \cup P_{m-3})}\right) + 1 = \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m-2}{3} \right\rfloor + 1.$$

This proves the proposition. $\square$

As a result of the previous proposition, we can prove the following corollary.

Corollary 7.33. *If* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 0, 1 \pmod 3$*, then*

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor$$

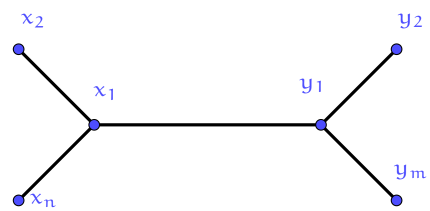
Proof. Note that $\lfloor \frac{k}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor + 1$ when $k \equiv 0, 1 \pmod 3$. From Theorem 7.27, in (7.3) the lower and upper bound coincide for these cases. So, the equality follows. $\square$

Now we have only three more cases left to deal with, i.e., the case $n \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$, the case $n \equiv 1 \pmod 3$, $m \equiv 2 \pmod 3$, and the case $n \equiv 2 \pmod 3$, $m \equiv 2 \pmod 3$.

Lemma 7.34. *If* $n \equiv 2 \pmod 3$ *and* $m \equiv 2 \pmod 3$*, then*

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) = \nu(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. We divide the graph into three subgraphs H_1, H_2 and H_3. We make $H_1 = C_n \setminus \{x_1\}$ and $H_2 = C_m \setminus \{y_1\}$. The subgraph H_3 is defined by taking the bridge $e = x_1y_1$ and the neighboring vertices $\{x_2, x_n, y_2, y_m\}$, i.e. the graph below.



Using this decomposition and Theorem 7.12 we get the inequality

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) \leqslant \operatorname{reg}(R/I(H_1)) + \operatorname{reg}(R/I(H_2)) + \operatorname{reg}(R/I(H_3)),$$

where H_1 and H_2 are paths of length $n-1$ and $m-1$, respectively, and using Theorem 7.15 we get

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) \leqslant \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Finally, in the present case $n \equiv 2 \pmod 3$ and $m \equiv 2 \pmod 3$ we have the equality $\nu(C_n \cdot P_2 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 1$, and so the proof follows from Theorem 7.16. □

Lemma 7.35. *If* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$*, then*

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_2 \cdot C_m)}\right) = \nu(C_n \cdot P_2 \cdot C_m) + 1 = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

Proof. In this case we delete the vertex x_1 from the cycle C_n. We have that $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ is an induced subgraph of $C_n \cdot P_2 \cdot C_m$ which is given as the disjoint union of a path of length $n-1$ and C_m, i.e. $H = P_{n-1} \cup C_m$. From Theorem 7.7(i), Corollary 7.14, Theorem 7.15 and Theorem 7.20 we get that

$$\operatorname{reg}(R/I(C_n \cdot P_2 \cdot C_m)) \geqslant \operatorname{reg}(R/I(H)) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

It follows from Proposition 7.32 and the fact that $\lfloor k/3 \rfloor = \lfloor (k-2)/3 \rfloor + 1$ when $k \equiv 0, 1 \pmod 3$ that

$$\operatorname{reg} R/I(C_n \cdot P_2 \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

So, the proof follows. □

Theorem 7.36. *Let* $n, m \geqslant 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_2 \cdot C_m) + 2 & \text{if } n \equiv 0, 1 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_2 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. It follows from Corollary 7.33, Lemma 7.34 and Lemma 7.35. □

The case $l = 3$

Throughout this subsection, we consider the dumbbell graph $C_n \cdot P_3 \cdot C_m$.

Proposition 7.37. *Let* $n, m \geqslant 3$*, then*

(i) $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \leqslant \nu(C_n \cdot P_3 \cdot C_m) + 2$, *if* $n, m \equiv 2 \pmod 3$*;*

(ii) $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) = \nu(C_n \cdot P_3 \cdot C_m) + 1$, *otherwise.*

Proof. Let $E(P_3) = \{e, e'\}$ be the set of edges of P_3, where $e = z_1z_2$ and $e' = z_2z_3$ are connected to C_n and C_m, respectively. Note that $I((C_n \cdot P_3 \cdot C_m) \setminus e) = I(C_n \cup (e' \cdot C_m))$ and that $I((C_n \cdot P_3 \cdot C_m)_e) = I(P_{n-3} \cup P_{m-1})$ because

$$N_G[e] = \{x_1 = z_1, x_2, x_n, z_2, y_1 = z_3\},$$

where $e' \cdot C_m$ is the unicyclic graph with C_m and a whisker e' attached to C_m, P_{n-3} is the path on the vertices $\{x_3, \ldots, x_{n-1}\}$ and P_{m-1} is the path on the vertices $\{y_2, \ldots, y_m\}$. Thus, Theorem 7.7(iii) gives the inequality

$$\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) \leqslant \max\left\{\operatorname{reg}\left(\frac{R}{I(C_n \cup (e' \cdot C_m))}\right), \operatorname{reg}\left(\frac{R}{I(P_{n-3} \cup P_{m-1})}\right) + 1\right\}.$$

From Proposition 7.26 and [4, Lemma 3.2] follows that $\operatorname{reg}(I(e' \cdot C_m)) = \lfloor \frac{m}{3} \rfloor + \lfloor \frac{3-\xi_3(m)}{3} \rfloor + 1$. Thus, using Remark 7.31, Corollary 7.14 and Theorem 7.15, we get $\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) \leqslant \max\left\{\left\lfloor \frac{n-2}{3} \right\rfloor + 1 + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{3-\xi_3(m)}{3} \right\rfloor, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + 1\right\}$.

On the other hand, from Theorem 7.27 we have that

$$\nu(C_n \cdot P_3 \cdot C_m) = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{4 - \xi_3(n) - \xi_3(m)}{3} \rfloor.$$

Therefore, we see that $\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) \leqslant \nu(C_n \cdot P_3 \cdot C_m) + 1$ when $n, m \equiv 2 \pmod 3$, and that $\operatorname{reg}\left(\frac{R}{I(C_n \cdot P_3 \cdot C_m)}\right) = \nu(C_n \cdot P_3 \cdot C_m)$ in all the remaining cases. □

Theorem 7.38. *Let* $n, m \geqslant 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) = \begin{cases} \nu(C_n \cdot P_3 \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod 3, \\ \nu(C_n \cdot P_3 \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. From Proposition 7.37, it suffices to show that $\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \geqslant \nu(C_n \cdot P_3 \cdot C_m) + 2$ when $n, m \equiv 2 \pmod 3$. Hence, we assume $n, m \equiv 2 \pmod 3$. Let z_2 be the middle vertex of $C_n \cdot P_3 \cdot C_m$. By removing z_2 we see that $H = (C_n \cdot P_3 \cdot C_m) \setminus z_2 = C_n \cup C_m$ is an induced subgraph of $C_n \cdot P_3 \cdot C_m$. From Theorem 7.20 and Corollary 7.14, we have that

$$\operatorname{reg} I(H) = \operatorname{reg} I(C_n) + \operatorname{reg} I(C_m) - 1 = \nu(C_n) + \nu(C_m) + 3.$$

Since $\nu(C_n \cdot P_3 \cdot C_m) = \nu(C_n) + \nu(C_m) + 1$, by using Theorem 7.7$(i)$ we get

$$\operatorname{reg} I(C_n \cdot P_3 \cdot C_m) \geqslant \operatorname{reg} I(H) = \nu(C_n \cdot P_3 \cdot C_m) + 2. \qquad \square$$

Regularity of a dumbbell graph

Now we are ready for the main result of this section. In the following theorem we compute the regularity of the edge ideal of the dumbbell graph $C_n \cdot P_l \cdot C_m$.

Theorem 7.39. *Let* $m, n \geqslant 3$ *and* $l \geqslant 1$*, then*

(i) if $l \equiv 0, 1 \pmod 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod 3, \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) if $l \equiv 2 \pmod 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & n \equiv 0, 1 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

Proof. Follows from Proposition 7.28, and Theorem 7.30, Theorem 7.36, and Theorem 7.38. $\square$

7.3 Combinatorial characterization of $\operatorname{reg}(I(G))$ in terms of $\nu(G)$

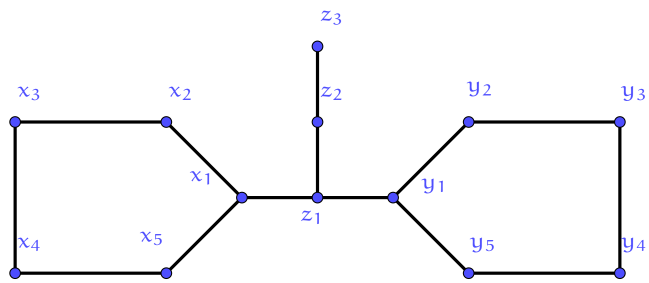
In this section, we focus on any bicyclic graph which admits the dumbbell graph $C_n \cdot P_l \cdot C_m$ as its base bicycle and study the regularity of the edge ideals of these graphs.

Let G be any bicyclic graph with dumbbell graph $C_n \cdot P_l \cdot C_m$. Then its decycling number is smaller or equal than 2. Thus by Theorem 7.16 and Theorem 7.18, we get

$$\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 3.$$

There are examples of bicyclic graphs with a dumbbell where the regularity of its edge ideal is equal to $\nu(G) + 1$, $\nu(G) + 2$ and $\nu(G) + 3$.

Example 7.40. *The following graph* G



has $\operatorname{reg} I(G) = 6$ *and* $\nu(G) = 3$.

In this section, we give a combinatorial characterization of the bicyclic graphs with regularity $\nu(G) + 1$, $\nu(G) + 2$ and $\nu(G) + 3$.

For the rest of the chapter, we shall use the term "dumbbell" of the bicyclic graph G, and it denotes the unique subgraph of G of the form $C_n \cdot P_l \cdot C_m$.

The following simple remark will be crucial in our treatment.

Remark 7.41. [4, Observation 2.1] *Let* G *be a graph with a leaf* y *and its unique neighbor* x, *say* $e = \{x, y\}$. *If* $\{e_1, \dots, e_s\}$ *is an induced matching in* $G \setminus N[x]$, *then* $\{e_1, \dots, e_s, e\}$ *is an induced matching in* G. *So we have* $\nu(G \setminus N[x]) + 1 \leqslant \nu(G)$.

Proposition 7.42. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$. *The following statements hold.*

(i) When $n, m \equiv 0, 1 \pmod 3$, *we have* $\operatorname{reg} I(G) = \nu(G) + 1$.

(ii) When $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$, *we have* $\operatorname{reg} I(G) \leqslant \nu(G) + 2$.

(iii) When $l \leqslant 2$, *we have* $\operatorname{reg} I(G) \leqslant \nu(G) + 2$.

Proof. (i) Again, it is enough to prove the upper bound $\operatorname{reg} I(G) \leqslant \nu(G) + 1$. Let E' be the set of edges $E' = E(G) \setminus E(C_n \cdot P_l \cdot C_m)$. We proceed by induction on the cardinality of E'. If $|E'| = 0$

then the statement follows from Theorem 7.39, so we assume $|E'| > 0$. There exists a leaf y in G such that $N[y] = \{x\}$. Let $G' = G \setminus x$ and $G'' = G \setminus N[x]$, then by Theorem 7.7 we have

$$\operatorname{regI}(G) \leqslant \max\{\operatorname{regI}(G'), \operatorname{regI}(G'') + 1\}.$$

The graphs G' and G'' can be either bicyclic graphs with the same dumbbell $C_n \cdot P_l \cdot C_m$, or the disjoint union of two unicyclic graphs with cycles C_n and C_m, or unicyclic graphs with a cycle C_r ($r = n$ or $r = m$) of the type $r \equiv 0, 1 \pmod 3$, or forests. Using either the induction hypothesis, or [4, Theorem 1.2] and Corollary 7.14, or [4, Theorem 1.2], or Theorem 7.15, then we get $\operatorname{regI}(G') = \nu(G') + 1$ and $\operatorname{regI}(G'') = \nu(G'') + 1$. Since we have $\nu(G') \leqslant \nu(G)$ and $\nu(G'') + 1 \leqslant \nu(G)$ (by Remark 7.41), we obtain the required inequality.

(ii) and (iii) follow by the same inductive argument, only changing the fact that G' and G'' could be unicyclic graphs with cycle C_r of the type $r \equiv 2 \pmod 3$. □

Remark 7.43. *The inductive process of the previous proposition cannot conclude* $\operatorname{regI}(G) \leqslant \nu(G) + 2$ *in the case* $l \geqslant 3$*. Here we may encounter two disjoint induced subgraphs* G_1 *and* G_2 *with* $\operatorname{regI}(G_i) = \nu(G_i) + 2$*, which implies* $\operatorname{regI}(G_1 \cup G_2) = \nu(G_1 \cup G_2) + 3$*. This is exactly the case of Example 7.40.*

An alternative proof of the inequality $\operatorname{regI}(G) \leqslant \nu(G) + 3$ *for* $l \geqslant 3$ *can be given by using the same inductive technique of Proposition 7.42.*

For the rest of the chapter we use the following notation.

Definition 7.44. *Let* G *be a graph,* $H \subset G$ *be a subgraph, and* v *and* w *be vertices of* G*. Then, we assume the following:*

(i) $d(v, w)$ *denotes the length (i.e., the number of edges) of a minimal path between* v *and* w*. In particular,* $d(v, v) = 0$*.*

(ii) $d(v, H)$ *denotes the minimal distance from the vertex* v *to the subgraph* H*, that is*

$$d(v, H) = \min\{d(v, w) \mid w \in H\}.$$

In particular, $d(v, H) = 0$ *if and only if* $v \in H$*.*

(iii) Let $H' \subset G$ *be a subgraph, then the distance between* H *and* H' *is given by*

$$d(H, H') = \min\{d(v, H') \mid v \in H\}.$$

In particular, $d(H, H') = 0$ *if and only if* $H \cap H' \neq \emptyset$*.*

(iv) $\Gamma_G(H)$ *denotes the subset of vertices*

$$\Gamma_G(H) = \{v \in G \mid d(v, H) = 1\}.$$

(v) *In the case* $k > 0$, $S_{G,k}(H)$ *denotes the induced subgraph given by restricting to the vertex set*

$$V(S_{G,k}(H)) = \{v \in G \mid d(v,H) \geqslant k\}.$$

(vi) $S_{G,0}(H)$ *denotes the subgraph given by the vertex set*

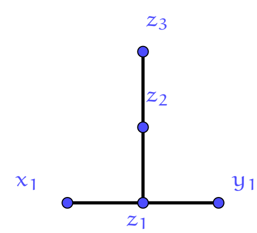
$$V(S_{G,0}(H)) = \{v \in G \mid d(v,H) > 0 \textit{ or } \deg(v) \geqslant 3\}.$$

and the edge set

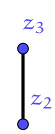
$$E(S_{G,0}(H)) = \{(v,w) \in E(G) \mid v,w \in V(S_{G,0}(H))\} \setminus \{(v,w) \in E(G) \mid v,w \in H\}.$$

We clarify the previous definition in the following example.

Example 7.45. *(i) Let* G *be the graph of Example 7.40 and* $H = C_5 \cup C_5$ *be the subgraph given by the two cycles of length* 5. *Then, we have that* $\Gamma_G(H)$ *is the set containing the vertex in the middle of the bridge joining the two cycles, that* $S_{G,0}(H)$ *is a graph of the form*

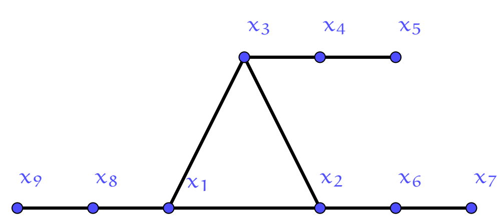


and that the graph

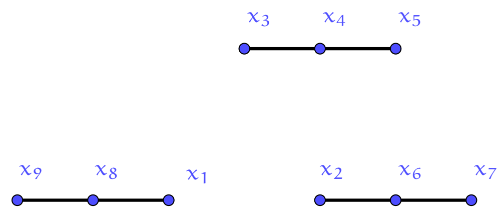


represents $S_{G,2}(H)$.

(ii) Let G *be the graph given by*



and H *be the triangle induced by the vertices* $\{x_1, x_2, x_3\}$. *Then, we have that* $\Gamma_G(H) = \{x_4, x_6, x_8\}$*, that* $S_{G,0}(H)$ *is a graph of the form*



and that the graph

represents $S_{G,2}(H)$.

We have already computed reg$I(G)$ in the case $n, m \equiv 0, 1 \pmod 3$, for the remaining cases we divide this section into subsections.

Case I

In this subsection we focus on the case $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. This case turns out to be almost identical to a unicyclic graph, and our treatment is influenced by [4, Section 3].

Setup 7.46. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$ *such that* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$*. We denote by* $F_1, \ldots, F_c$ *the connected components of* $S_{G,0}(C_m)$*, and in this case each* F_i *is either a tree or a unicyclic graph with cycle* C_n *(and* $n \equiv 0, 1 \pmod 3$*). Then, the graph* $S_{G,2}(C_m)$ *can be given as the union of the components* $H_1, \ldots, H_c$*, where each one is defined as*

$$H_i = F_i \setminus \{v \in G \mid d(v, C_m) \leqslant 1\}.$$

Note that each H_i *can be a disconnected graph or even the empty graph.*

Remark 7.47. *The following statements hold.*

(i) The graph $G \setminus \Gamma_G(C_m)$ *has a decomposition of the form*

$$G \setminus \Gamma_G(C_m) = C_m \bigcup \left(\bigcup_{i=1}^{c} H_i \right),$$

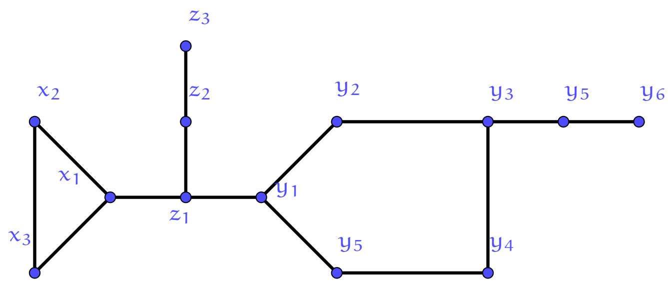
and in particular

$$\nu\left(G \setminus \Gamma_G(C_m)\right) = \nu\left(C_m\right) + \sum_{i=1}^{c} \nu\left(H_i\right)$$

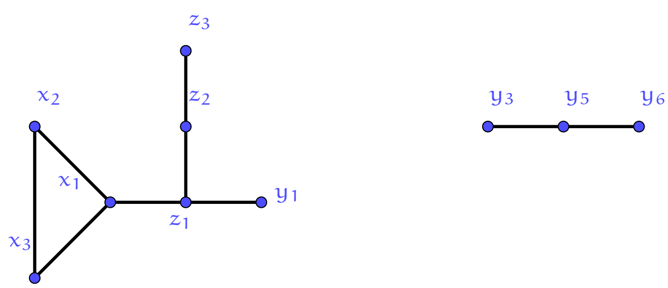
because $d(C_m, H_i) \geqslant 2$ *for all* $1 \leqslant i \leqslant c$ *and* $d(H_i, H_j) \geqslant 2$ *for all* $1 \leqslant i < j \leqslant c$.

(ii) For each $i = 1, \ldots, c$, *we have that* $|F_i \cap C_m| = 1$.

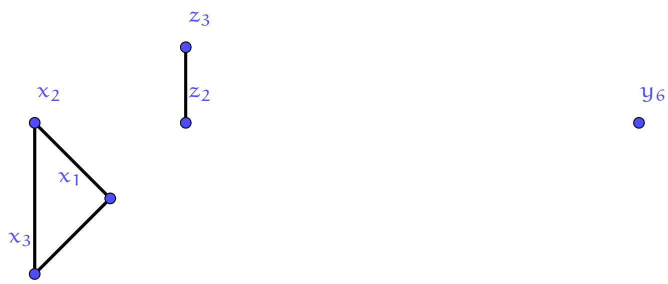
Example 7.48. *Let* G *be the graph*



and C_5 *be the cycle given by* $\{y_1, y_2, y_3, y_4, y_5\}$. *We have that* $\Gamma_G(C_5) = \{z_1, y_5\}$. *The graph* $S_{G,0}(C_5)$ *is given by*



with connected components F_1 *(graph on the left) and* F_2 *(graph on the right). The graph* $S_{G,2}(C_5)$ *is given by*



with connected components H_1 *(graph on the left) and* H_2 *(graph on the right).*

Lemma 7.49. *Adopt Setup 7.46. If* $\nu(H_i) = \nu(F_i)$ *for all* $1 \leqslant i \leqslant c$*, then* $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$.

Proof. Follows identically to [4, Lemma 3.5]. □

Proposition 7.50. *Adopt Setup 7.46. If* $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$ *then* $\operatorname{reg}I(G) = \nu(G) + 1$.

Proof. Once more, we shall only prove that $\operatorname{reg}I(G) \leqslant \nu(G) + 1$. Assume that $\nu(G \setminus \Gamma_G(C_m)) < \nu(G)$, then the contrapositive of Lemma 7.49 implies that there exists some i with $\nu(H_i) < \nu(F_i)$.

Fix i such that $\nu(H_i) < \nu(F_i)$. From Remark 7.47(ii), let x be the vertex in $F_i \cap C_m$. Let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Again, we have the inequality

$$\operatorname{reg}I(G) \leqslant \max\{\operatorname{reg}I(G'), \operatorname{reg}I(G'') + 1\}.$$

Note that both G' and G'' can be either unicyclic graphs with cycle C_n (and $n \equiv 0, 1 \pmod 3$), or forests. Hence, from [4, Theorem 1.2] and Theorem 7.15 we get that $\operatorname{reg}I(G') = \nu(G') + 1$ and $\operatorname{reg}I(G'') = \nu(G'') + 1$.

In the case of G', we have that $\operatorname{reg}I(G') = \nu(G') + 1 \leqslant \nu(G) + 1$. Let H be the induced subgraph of G obtained by deleting the vertices of $F_i \cup N_G[x]$. Then we have $G'' = H \cup H_i$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be maximal induced matchings in H and H_i, respectively, then $\nu(G'') = |\mathcal{M}_1| + |\mathcal{M}_2|$ because $d(H, H_i) \geqslant 2$. By the condition $\nu(F_i) > \nu(H_i)$ then there exists a maximal induced matching $\mathcal{M}_3$ in F_i, such that $|\mathcal{M}_3| > |\mathcal{M}_2|$. From the fact that $H \cup F_i$ is an induced subgraph in G and $d(H, F_i) \geqslant 2$, then we get

$$\nu(G) \geqslant \nu(H \cup F_i) = |\mathcal{M}_1| + |\mathcal{M}_3| > |\mathcal{M}_1| + |\mathcal{M}_2| = \nu(G'').$$

Hence $\operatorname{reg}I(G'') = \nu(G'') + 1 \leqslant \nu(G)$, and so we get the statement of the proposition. □

Theorem 7.51. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$ *such that* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$*. Then the following statements hold.*

(i) $\nu(G) + 1 \leqslant \operatorname{reg}I(G) \leqslant \nu(G) + 2$*;*

(ii) $\operatorname{reg}I(G) = \nu(G) + 2$ *if and only if* $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

Proof. In Proposition 7.42 we proved (i). In order to prove (ii), we only need to show that $\nu(G \setminus \Gamma_G(C_m)) = \nu(G)$ implies $\operatorname{reg}I(G) \geqslant \nu(G) + 2$, because the inverse implication follows from Proposition 7.50.

From Remark 7.47(i), $G \setminus \Gamma_G(C_m) = C_m \cup (\cup_{i=1}^{c} H_i)$ where each H_i is either a forest or a unicyclic graph with cycle C_n (and $n \equiv 0, 1 \pmod 3$). Then, from Corollary 7.14, [4, Theorem 1.2] and Theorem 7.15 we get

$$\begin{aligned}
\operatorname{reg}I(G \setminus \Gamma_G(C_m)) &= \operatorname{reg}I(C_m) + \operatorname{reg}I\left(\cup_{i=1}^{c} H_i\right) - 1 \\
&= (\nu(C_m) + 2) + \left(\nu\left(\cup_{i=1}^{c} H_i\right) + 1\right) - 1 \\
&= \nu(G \setminus \Gamma_G(C_m)) + 2 \\
&= \nu(G) + 2.
\end{aligned}$$

Finally, since $G \setminus \Gamma_G(C_m)$ is an induced subgraph of G then we have $\operatorname{reg}I(G) \geqslant \nu(G) + 2$. $\square$

Case II

The object of study of this subsection is the case where $n, m \equiv 2 \pmod 3$, $l \geqslant 3$, and in particular when $\operatorname{reg}I(G) = \nu(G) + 3$. More specifically, we shall give necessary and sufficient conditions for the equality $\operatorname{reg}I(G) = \nu(G) + 3$.

Setup 7.52. *Let G be a bicyclic graph with dumbbell graph $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geqslant 3$. As in Setup 7.46, let $F_1, \ldots, F_c$ be the components of the graph $S_{G,0}(C_n)$. We order the F_i's in such a way that F_1 is a unicyclic graph with cycle C_m, and for all $i > 1$ we have that F_i is a tree. The graph $S_{G,2}(C_n)$ can be decomposed in components $H_1, \ldots, H_c$ where*

$$H_i = F_i \setminus \{v \in G \mid d(v, C_n) \leqslant 1\}.$$

Remark 7.53. *From the previous setup we get the following simple remarks.*

(i) The graph $G \setminus \Gamma_G(C_n)$ has a decomposition of the form

$$G \setminus \Gamma_G(C_n) = C_n \bigcup \left(\bigcup_{i=1}^{c} H_i \right),$$

and in particular

$$\nu\left(G \setminus \Gamma_G(C_n)\right) = \nu\left(C_n\right) + \sum_{i=1}^{c} \nu\left(H_i\right)$$

because $d(C_n, H_i) \geqslant 2$ for all $1 \leqslant i \leqslant c$ and $d(H_i, H_j) \geqslant 2$ for all $1 \leqslant i < j \leqslant c$.

(ii) *Similarly, the graph* $G \setminus \Gamma_G(C_n \cup C_m)$ *has a decomposition of the form*

$$G \setminus \Gamma_G(C_n \cup C_m) = C_n \bigcup \left(\bigcup_{i=2}^{c} H_i \right) \bigcup (H_1 \setminus \Gamma_{H_1}(C_m)),$$

and in particular

$$\nu(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)).$$

(iii) *For each* $i = 1, \ldots, c$*, we have that* $|F_i \cap C_n| = 1$*.*

(iv) *The statement of Lemma 7.49 also holds in this case, that is, if* $\nu(H_i) = \nu(F_i)$ *for all* $1 \leqslant i \leqslant c$*, then* $\nu(G \setminus \Gamma_G(C_n)) = \nu(G)$*.*

(v) *Due to the assumption* $l \geqslant 3$*, then we have that* C_m *must be an induced subgraph of* H_1*. During this subsection and the next one we shall fundamentally use this fact, and it will allow us to inductively "separate" the two cycles* C_n *and* C_m*.*

Lemma 7.54. *Adopt Setup 7.52. If* $\nu(H_i) = \nu(F_i)$ *for all* $1 \leqslant i \leqslant c$ *and* $\nu(H_1) = \nu(H_1 \setminus \Gamma_{H_1}(C_m))$*, then*

$$\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G).$$

Proof. Since $G \setminus \Gamma_G(C_n \cup C_m)$ is an induced subgraph of G, we have $\nu(G \setminus \Gamma_G(C_n \cup C_m)) \leqslant \nu(G)$. From Remark 7.53(ii) we get

$$\begin{aligned}
\nu(G \setminus \Gamma_G(C_n \cup C_m)) &= \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)) \\
&= \nu(C_n) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1) \\
&= \nu(C_n) + \sum_{i=1}^{c} \nu(F_i) \\
&\geqslant \nu(G),
\end{aligned}$$

and so $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$. □

Proposition 7.55. *Adopt Setup 7.52. If* $\nu(G \setminus \Gamma_G(C_n \cup C_m)) < \nu(G)$*, then*

$$\operatorname{reg} I(G) \leqslant \nu(G) + 2.$$

Proof. From the contrapositive of Lemma 7.54, it follows that there exists some i with $\nu(H_i) < \nu(F_i)$ or we have $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Then we divide the proof into two cases.

Case 1. In this case we assume that for some $1 \leqslant i \leqslant c$ we have $\nu(H_i) < \nu(F_i)$. This case follows similarly to Proposition 7.50. Let x be the vertex in $F_i \cap C_n$, let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Once more, we have the inequality

$$\mathrm{reg} I(G) \leqslant \max\{\mathrm{reg} I(G'), \mathrm{reg} I(G'') + 1\}.$$

Note that both G' and G'' are unicyclic graphs, and so we have $\mathrm{reg} I(G') \leqslant \nu(G')+2$ and $\mathrm{reg} I(G'') \leqslant \nu(G'') + 2$ (see Theorem 7.18). Since we have $\nu(G') \leqslant \nu(G)$ and $\nu(G'') + 1 \leqslant \nu(G)$ (see the proof of Proposition 7.50), the inequality follows in this case.

Case 2. Now we suppose that $\nu(H_1 \setminus \Gamma_{H_1}(C_m)) < \nu(H_1)$. Let x be the vertex in $F_1 \cap C_n$, let us use the notations $G' = G \setminus x$ and $G'' = G \setminus N[x]$. We use the inequality

$$\mathrm{reg} I(G) \leqslant \max\{\mathrm{reg} I(G'), \mathrm{reg} I(G'') + 1\}.$$

The graphs G' and G'' are unicyclic. For the graph G', we have $\mathrm{reg} I(G') \leqslant \nu(G') + 2 \leqslant \nu(G) + 2$.

The graph G'' can be given as the disjoint union of H_1 and another graph H defined by $H = G \setminus (F_1 \cup N[x])$, that is $G'' = H \cup H_1$ and $d(H, H_1) \geqslant 2$. Thus it follows that $\nu(G'') = \nu(H) + \nu(H_1)$ and that $\mathrm{reg} I(G'') = \mathrm{reg} I(H) + \mathrm{reg} I(H_1) - 1$ (see Corollary 7.14).

Since H is a forest, Theorem 7.15 gives $\mathrm{reg} I(H) = \nu(H) + 1$. From [4, Corollary 3.11], it follows that $\mathrm{reg} I(H_1) = \nu(H_1) + 1$. By summing up, we obtain that $\mathrm{reg} I(G'') \leqslant \nu(G'') + 1$. So we get the inequality $\mathrm{reg} I(G'') + 1 \leqslant \nu(G'') + 2 \leqslant \nu(G) + 2$, because G'' is an induced subgraph of G. □

Now we are ready to completely describe the case where $\mathrm{reg} I(G) = \nu(G) + 3$.

Theorem 7.56. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$. Then,* $\mathrm{reg} I(G) = \nu(G) + 3$ *if and only if the following conditions are satisfied:*

(i) $n \equiv 2 \pmod 3$;

(ii) $m \equiv 2 \pmod 3$;

(iii) $l \geqslant 3$;

(iv) $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$.

Proof. In Proposition 7.42 we proved that the conditions (i), (ii) and (iii) are necessary, and from Proposition 7.55 we have that the condition (iv) is also necessary. Hence, we only need to prove that $\mathrm{reg} I(G) = \nu(G) + 3$ under these conditions.

Let $W = G \setminus \Gamma_G(C_n \cup C_m)$. From Remark 7.53(ii) and Corollary 7.14, we obtain the equality

$$\mathrm{reg}\big(I(W)\big) = \mathrm{reg}\big(I(C_n)\big) + \mathrm{reg}\big(I\big(\cup_{i=2}^{c} H_i\big)\big) + \mathrm{reg}\big(I\big(H_1 \setminus \Gamma_{H_1}(C_m)\big)\big) - 2.$$

Note that the graph $H_1 \setminus \Gamma_{H_1}(C_m)$ can be given as the disjoint union of the cycle C_m and the forest $H = (H_1 \setminus \Gamma_{H_1}(C_m)) \setminus C_m$, such that $d(H, C_m) \geqslant 2$. From Theorem 7.20 and Theorem 7.15 we get $\operatorname{regI}(C_m) = \nu(C_m)+2$ and $\operatorname{regI}(H) = \nu(H)+1$, respectively, and so Corollary 7.14 implies that $\operatorname{regI}(H_1 \setminus \Gamma_{H_1}(C_m)) = \operatorname{regI}(C_m) + \operatorname{regI}(H) - 1 = \nu(C_m) + \nu(H) + 2 = \nu(H_1 \setminus \Gamma_{H_1}(C_m)) + 2$.

Therefore, by also using Theorem 7.20 and Theorem 7.15, we obtain

$$\begin{aligned}
\operatorname{reg}(I(W)) &= (\nu(C_n) + 2) + (\nu(\cup_{i=2}^{c} H_i) + 1) + (\nu(H_1 \setminus \Gamma_{H_1}(C_m)) + 2) - 2 \\
&= \nu(W) + 3 \\
&= \nu(G) + 3.
\end{aligned}$$

Since W is an induced subgraph of G then we get

$$\operatorname{regI}(G) \geqslant \operatorname{regI}(W)) = \nu(G) + 3,$$

and so from Theorem 7.18 the equality is obtained. □

Case III

In this subsection we assume that G is a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geqslant 3$. Now that we have characterized when $\operatorname{regI}(G) = \nu(G) + 3$, then we want to distinguish between $\operatorname{regI}(G) = \nu(G) + 1$ and $\operatorname{regI}(G) = \nu(G) + 2$.

Lemma 7.57. *Adopt Setup 7.52. If* $\nu(G) - \nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = 1$ *then*

$$\operatorname{regI}(G) = \nu(G) + 2.$$

Proof. From Theorem 7.56 we have that $\operatorname{reg}(I(G)) \leqslant \nu(G) + 2$. Using the same method as in Theorem 7.56, we can obtain a lower bound

$$\operatorname{regI}(G) \geqslant \operatorname{regI}(G \setminus \Gamma_G(C_n \cup C_m)) = \nu(G \setminus \Gamma_G(C_n \cup C_m)) + 3 = \nu(G) + 2,$$

and so the equality follows. □

Lemma 7.58. *Adopt Setup 7.52. If* $\nu(G) = \nu(G \setminus \Gamma_G(C_n))$ *then*

$$\operatorname{regI}(G) \geqslant \nu(G) + 2.$$

Symmetrically, the same argument holds for C_m.

Proof. The proof follows similarly to Theorem 7.51. From Remark 7.53(i), Corollary 7.14, Theorem 7.20 and Theorem 7.16 we get

$$\begin{aligned}\operatorname{regI}(G \setminus \Gamma_G(C_n)) &= \operatorname{regI}(C_n) + \operatorname{regI}(\cup_{i=1}^{c} H_i) - 1\\ &\geqslant (\nu(C_n)+2) + (\nu(\cup_{i=1}^{c} H_i)+1) - 1\\ &\geqslant \nu(G \setminus \Gamma_G(C_n)) + 2\\ &\geqslant \nu(G)+2.\end{aligned}$$

So the inequality follows from the fact that $G \setminus \Gamma_G(C_n)$ is an induced subgraph of G. □

The following simple logical argument will be used several times in the next theorem.

Observation 7.59. *Let P_1, P_2, P_3 be boolean values, (i.e. true or false). Assume that P_1 is true if and only if P_2 and P_3 are true, that is*

$$P_1 \iff (P_2 \quad \wedge \quad P_3)\,.$$

Suppose that if P_2 is true then P_3 is false, that is

$$P_2 \Longrightarrow \neg P_3.$$

Then, P_1 is false.

Notation 7.60. *Let X be a mathematical expression. Then, $P[X]$ represents a boolean value, which is true if X is satisfied and false otherwise.*

Taking into account the induced matching numbers $\nu(G)$, $\nu(G\setminus\Gamma_G(C_n \cup C_m))$, $\nu(G\setminus\Gamma_G(C_n))$ and $\nu(G \setminus \Gamma_G(C_m))$, we can give necessary and sufficient conditions for the equality

$$\operatorname{regI}(G) = \nu(G)+1.$$

Theorem 7.61. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \geqslant 3$. Then $\operatorname{regI}(G) = \nu(G)+1$ if and only if the following conditions are satisfied:*

(i) $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;

(ii) $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;

(iii) $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

Proof. From Theorem 7.56, Lemma 7.57 and Lemma 7.58, we have that the conditions (i), (ii) and (iii) are necessary. Hence, it is enough to prove $\operatorname{regI}(G) \leqslant \nu(G)+1$ under these conditions.

For any $x \in G$ we denote $G' = G \setminus x$ and $G'' = G \setminus N[x]$. Then, we have the upper bound

$$\operatorname{regI}(G) \leqslant \max\{\operatorname{regI}(G'), \operatorname{regI}(G'')+1\}.$$

We shall prove that under the conditions (i), (ii) and (iii) there exists a vertex $x \in C_n$ such that $\operatorname{reg} I(G') \leqslant \nu(G) + 1$ and $\operatorname{reg} I(G'') + 1 \leqslant \nu(G) + 1$. We divide the proof into three steps.

Step 1. In this step we prove that for any $x \in C_n$ we have $\operatorname{reg} I(G') \leqslant \nu(G) + 1$. First we note the following two observations:

- It follows from Theorem 7.18 that $\operatorname{reg} I(G') \leqslant \nu(G') + 2$. Hence, $\nu(G') < \nu(G)$ implies that $\operatorname{reg} I(G') \leqslant \nu(G') + 2 \leqslant \nu(G) + 1$.
- Since G' is a unicyclic graph, [4, Theorem 1.2] implies that $\operatorname{reg} I(G') = \nu(G') + 2$ if and only if $\nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))$.

Thus, it follows that

$$\operatorname{reg} I(G') = \nu(G) + 2 \iff \Big(\nu(G) = \nu(G') \text{ and } \nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))\Big).$$

In Observation 7.59, let $P_1 = P\big[\operatorname{reg} I(G') = \nu(G) + 2\big]$, $P_2 = P\big[\nu(G) = \nu(G')\big]$ and $P_3 = \big[\nu(G') = \nu(G' \setminus \Gamma_{G'}(C_m))\big]$. From the logical argument of Observation 7.59, if we prove that $\nu(G') = \nu(G)$ implies $\nu(G') > \nu(G' \setminus \Gamma_{G'}(C_m))$ then we get the desired inequality $\operatorname{reg} I(G') \leqslant \nu(G) + 1$. Assume that $\nu(G) = \nu(G')$. From the hypothesis $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$ and the fact that $G' \setminus \Gamma_{G'}(C_m)$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$, we get

$$\nu(G') = \nu(G) > \nu(G \setminus \Gamma_G(C_m)) \geqslant \nu(G' \setminus \Gamma_{G'}(C_m)).$$

Therefore, we have $\operatorname{reg} I(G') \leqslant \nu(G) + 1$.

Step 2. Since $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$, it follows from Remark 7.53(iv) that there exists some $1 \leqslant i \leqslant c$ such that $\nu(F_i) > \nu(H_i)$. Following Setup 7.52, we have that F_1 is a unicyclic graph containing the cycle C_m and that F_i is a tree for all $i > 1$. In this step, fix $i > 1$ where F_i is a tree and $\nu(F_i) > \nu(H_i)$.

Let x be the vertex in $F_i \cap C_n$ and H be the induced subgraph $H = G \setminus (F_i \cup N[x])$. Note that $G'' = H \cup H_i$, $d(H, H_i) \geqslant 2$ and $d(H, F_i) \geqslant 2$. Then

$$\nu(G'') = \nu(H) + \nu(H_i) < \nu(H) + \nu(F_i) \leqslant \nu(G)$$

follows from the condition $\nu(H_i) < \nu(F_i)$. So we have that $\nu(G'') < \nu(G)$.

As in Step 1, we note the following two observations:

- It follows from Theorem 7.18 that $\operatorname{reg} I(G'') \leqslant \nu(G'') + 2$. Hence, $\nu(G'') + 1 < \nu(G)$ implies that $\operatorname{reg} I(G'') + 1 \leqslant \nu(G'') + 3 \leqslant \nu(G) + 1$.
- Since G'' is a unicyclic graph, [4, Theorem 1.2] implies that $\operatorname{reg} I(G'') = \nu(G'') + 2$ if and only if $\nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))$.

So, we have that

$$\operatorname{reg} I(G'') + 1 = \nu(G) + 2 \iff \Big(\nu(G) = \nu(G'') + 1 \text{ and } \nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))\Big).$$

Let K be the induced subgraph defined by $K = (G \setminus \Gamma_G(C_m)) \setminus (F_i \cup N[x])$. Since $i > 1$ then

$F_i \cap F_1 = \emptyset$, and so we get the following statements:

- $G'' \setminus \Gamma_{G''}(C_m) = K \cup H_i$, because $G'' = H \cup H_i$ where $H = G \setminus (F_i \cup N[x])$, $C_m \subset H$ and $d(C_m, H_i) \geqslant 2$.
- $K \cup F_i$ is an induced subgraph of $G \setminus \Gamma_G(C_m)$.
- Since $d(K, F_i) \geqslant 2$ and $d(K, H_i) \geqslant 2$, we have the following inequalities

$$\nu(G'' \setminus \Gamma_{G''}(C_m)) = \nu(K) + \nu(H_i) < \nu(K) + \nu(F_i) \leqslant \nu(G \setminus \Gamma_G(C_m)).$$

In Observation 7.59, let $P_1 = P\big[\operatorname{reg}I(G'') + 1 = \nu(G) + 2\big]$, $P_2 = P\big[\nu(G) = \nu(G'') + 1\big]$ and $P_3 = \big[\nu(G'') = \nu(G'' \setminus \Gamma_{G'}(C_m))\big]$. So, it is enough to prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$. Assuming $\nu(G) = \nu(G'') + 1$, we get

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_m)) - 1 \geqslant \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Therefore, in this case we have $\operatorname{reg}I(G'') + 1 \leqslant \nu(G) + 1$.

Step 3. In this last step we assume that $\nu(F_1) > \nu(H_1)$ and that $\nu(F_i) = \nu(H_i)$ for all $i > 1$. Let x be the vertex in $F_1 \cap C_n$, then as in Step 2 we have that:

- $\nu(G'') < \nu(G)$.
- $\operatorname{reg}I(G'') + 1 = \nu(G) + 2 \iff \Big(\nu(G) = \nu(G'') + 1 \text{ and } \nu(G'') = \nu(G'' \setminus \Gamma_{G''}(C_m))\Big)$.

Once more, if we prove that $\nu(G) = \nu(G'') + 1$ implies $\nu(G'') > \nu(G'' \setminus \Gamma_{G''}(C_m))$ then we obtain that $\operatorname{reg}I(G'') + 1 \leqslant \nu(G) + 1$.

We denote by L the induced subgraph of $G'' \setminus \Gamma_{G''}(C_m)$ given by

$$L = (G'' \setminus \Gamma_{G''}(C_m)) \setminus \Gamma_G(C_n).$$

Due to Remark 7.53(ii), the graph L has the decomposition

$$L = (C_n \setminus N_{C_n}[x]) \bigcup \left(\bigcup_{i=1}^{c} H_i\right) \bigcup (H_1 \setminus \Gamma_{H_1}(C_m)),$$

with all the disjoint components at distance at least two between each other, and so we have

$$\nu(L) = \nu\left(C_n \setminus N_{C_n}[x]\right) + \sum_{i=2}^{c} \nu(H_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)).$$

By proceeding as in the proofs of Lemma 7.49 or Lemma 7.54, from the conditions $\nu(F_i) = \nu(H_i)$ for all $i > 1$, we obtain

$$\begin{aligned}\nu(L) &= \nu\left((C_n \setminus N_{C_n}[x])\right) + \sum_{i=2}^{c} \nu(F_i) + \nu(H_1 \setminus \Gamma_{H_1}(C_m)) \\ &\geqslant \nu\left(G'' \setminus \Gamma_{G''}(C_m)\right).\end{aligned}$$

Thus, $\nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m))$ because L is an induced subgraph of $G'' \setminus \Gamma_{G''}(C_m)$. We also have that L is an induced subgraph of $G \setminus \Gamma_G(C_n \cup C_m)$ because we have the equality

$$L = (G \setminus \Gamma_G(C_n \cup C_m)) \setminus N[x].$$

Finally, from the hypothesis $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$ we obtain

$$\nu(G'') = \nu(G) - 1 > \nu(G \setminus \Gamma_G(C_n \cup C_m)) \geqslant \nu(L) = \nu(G'' \setminus \Gamma_{G''}(C_m)).$$

Therefore, in this case we also have $\operatorname{regI}(G'') + 1 \leqslant \nu(G) + 1$, and so the proof follows. $\square$

Case IV

In this short subsection we deal with the remaining case, we assume that G is a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$ such that $n, m \equiv 2 \pmod 3$ and $l \leqslant 2$.

When $l \leqslant 2$, the two cycles are too close to each other, and it is difficult to make a direct analysis (with our methods). Fortunately, using the complete characterization of the case $l \geqslant 3$, the problem can be solved with the Lozin transformation (Definition 7.23).

Construction 7.62. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$. *Suppose that* x *is a vertex on the bridge* P_l, *then we apply the Lozin transformation of* G *with respect to* x *as follows:*

(i) *let* G_1 *and* G_2 *be the two connected components of* $(C_n \cdot P_l \cdot C_m) \setminus x$;

(ii) *partition the neighborhood* $N_G(x)$ *of the vertex* x *into two subsets* Y *and* Z *in a way such that* $Y \cap (C_n \cdot P_l \cdot C_m) \subset G_1$ *and* $Z \cap (C_n \cdot P_l \cdot C_m) \subset G_2$;

(iii) *delete vertex* x *from the graph together with its incident edges;*

(iv) *add a path* $P_4 = (y, a, b, z)$ *to the rest of the graph;*

(v) *connect the vertex* y *of the* P_4 *to each vertex in* Y, *and connect* z *to each vertex in* Z.

Then, we obtain a bicyclic graph, denoted by $\mathcal{L}_x(G)$, *with dumbbell* $C_n \cdot P_{l+3} \cdot C_m$.

From [107, Lemma 1] and [11, Theorem 1.1] we get the equality

$$\operatorname{reg}(I(\mathcal{L}_x(G))) - \nu(\mathcal{L}_x(G)) = \operatorname{reg}(I(G)) - \nu(G). \tag{7.4}$$

Therefore we obtain a characterization in the following corollary.

Corollary 7.63. *Let* G *be a bicyclic graph with dumbbell* $C_n \cdot P_l \cdot C_m$ *such that* $n, m \equiv 2 \pmod 3$ *and* $l \leqslant 2$. *Let* x *be a point on the bridge* P_l *and let* $\mathcal{L}_x(G)$ *be the Lozin transformation of* G *with respect to* x *given as in Construction 7.62. Then we have that* $\nu(G) + 1 \leqslant \operatorname{regI}(G) \leqslant \nu(G) + 2$, *and that* $\operatorname{regI}(G) = \nu(G) + 1$ *if and only if the following conditions are satisfied:*

(i) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;

(ii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;

(iii) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. From Proposition 7.42, it follows that $\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 2$. Due to (7.4), we can apply the Lozin transformation and reduce the problem to the case where the bridge has more than three vertices. Finally, Theorem 7.61 gives us the result. □

The characterization

Finally, the theorem below contains the characterization that we have found.

Theorem 7.64. *Let G be a bicyclic graph with dumbbell $C_n \cdot P_l \cdot C_m$.*

(I) *If* $n, m \equiv 0, 1 \pmod 3$, *then* $\operatorname{reg} I(G) = \nu(G) + 1$.

(II) *If* $n \equiv 0, 1 \pmod 3$ *and* $m \equiv 2 \pmod 3$, *then*

$$\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 2,$$

and $\operatorname{reg} I(G) = \nu(G) + 2$ *if and only if* $\nu(G) = \nu(G \setminus \Gamma_G(C_m))$.

(III) *If* $n, m \equiv 2 \pmod 3$ *and* $l \geqslant 3$, *then* $\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 3$. *Moreover:*

- *(i)* $\operatorname{reg} I(G) = \nu(G) + 3$ *if and only if* $\nu\left(G \setminus \Gamma_G(C_n \cup C_m)\right) = \nu(G)$.
- *(ii)* $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if the following conditions hold:*
  - *(a)* $\nu(G) - \nu(G \setminus \Gamma_G(C_n \cup C_m)) > 1$;
  - *(b)* $\nu(G) > \nu(G \setminus \Gamma_G(C_n))$;
  - *(c)* $\nu(G) > \nu(G \setminus \Gamma_G(C_m))$.

(IV) *If* $n, m \equiv 2 \pmod 3$ *and* $l \leqslant 2$, *then* $\nu(G) + 1 \leqslant \operatorname{reg} I(G) \leqslant \nu(G) + 2$. *Let x be a point on the bridge P_l and let $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to x given as in Construction 7.62. Then,* $\operatorname{reg} I(G) = \nu(G) + 1$ *if and only if the following conditions are satisfied:*

(a) $\nu(\mathcal{L}_x(G)) - \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n \cup C_m)) > 1$;

(b) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_n))$;

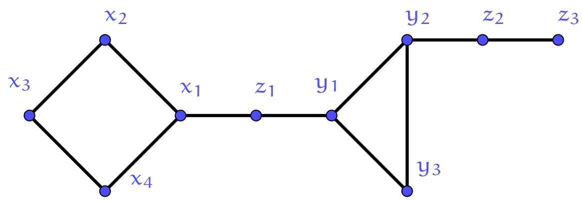
(c) $\nu(\mathcal{L}_x(G)) > \nu(\mathcal{L}_x(G) \setminus \Gamma_{\mathcal{L}_x(G)}(C_m))$.

Proof. Statement *(I)* follows from Proposition 7.42. In Theorem 7.51, *(II)* is proved. By Theorem 7.56 and Theorem 7.61, we get *(III)*. Finally, from Corollary 7.63, we obtain *(IV)*. □

Examples

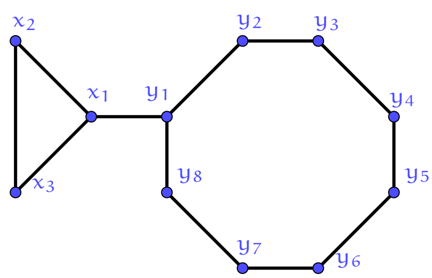
In this last subsection we give examples for each one of the statements in the characterization of Theorem 7.64.

Example 7.65 (Statement *(I)* of Theorem 7.64)**.** *Let* G *be the graph:*



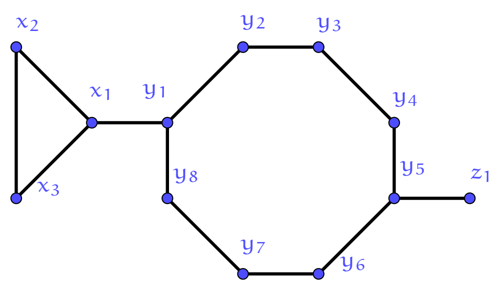
Then, we have $\mathrm{reg} I(G) = 4$ *and* $\nu(G) = 3$.

Example 7.66 (Statement *(II)* of Theorem 7.64)**.** *Let* G *be the graph:*



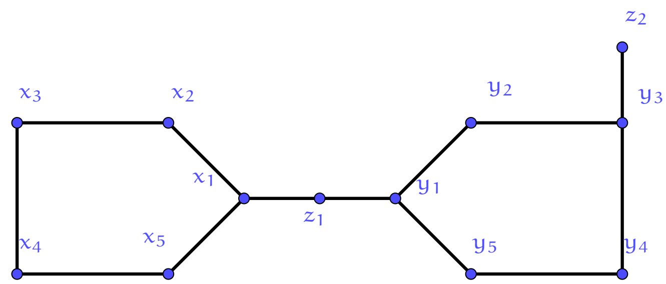
Then, we have $\mathrm{reg} I(G) = 5$ *and* $\nu(G) = 3$.

If G *is the graph given below, then we have* $\mathrm{reg} I(G) = 5$ *and* $\nu(G) = 4$.



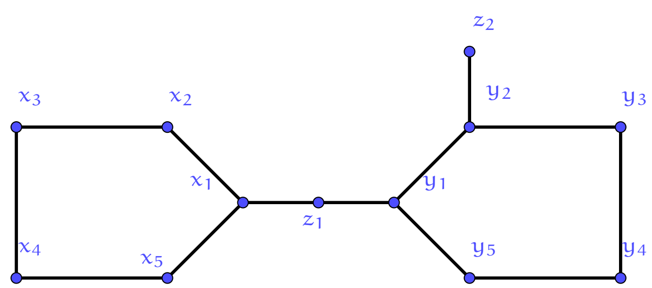
Example 7.67 (Statement *(III)* of Theorem 7.64)**.** *Let* G *be the graph given in Example 7.40. Then, we have* $\mathrm{reg} I(G) = 6$ *and* $\nu(G) = 3$.

Let G *be the graph:*



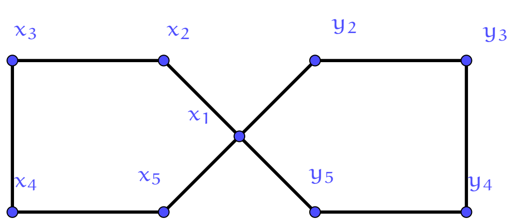
Then, we have reg$I(G) = 5$ *and* $\nu(G) = 3$.

Let G *be the graph given below and obtained by moving the whisker to the left.*



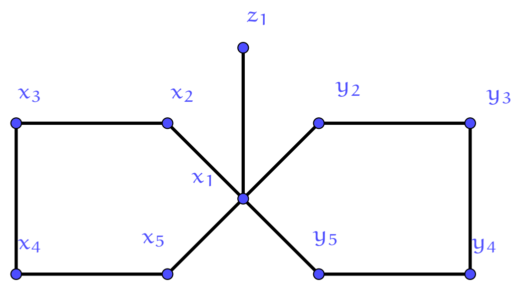
Then, we have reg$I(G) = 5$ *and* $\nu(G) = 4$.

Example 7.68 (Statement *(IV)* of Theorem 7.64)**.** *Let* G *be the graph:*



Then, we have reg$I(G) = 4$ *and* $\nu(G) = 2$.

Let G *be the graph given below and obtained by adding a whisker to the above graph at the join vertex* x_1.



Then, we have $\operatorname{reg} I(G) = 4$ *and* $\nu(G) = 3$.

7.4 Castelnuovo-Mumford regularity of powers

In this section, we study the regularity of the powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leqslant 2$. Our strategy to compute $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ for $q \geqslant 1$ relies on finding an upper bound and a lower bound on $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ where these bounds coincide and are equal to

$$2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2.$$

In order to obtain an upper bound, we follow the even-connection argument given in [8, Theorem 5.2]. Then, we proceed by looking at "nice" induced subgraphs of $C_n \cdot P_l \cdot C_m$ and we find a lower bound on $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ which is equal to the found upper bound.

Let I be an arbitrary ideal generated in degree d and let $b_q := \operatorname{reg}(I^q) - dq$ for $q \geqslant 1$. An interesting question is to study of the sequence $\{b_i\}_{i \geqslant 1}$. In [48] Eisenbud and Harris proved that if $\dim(R/I) = 0$, then $\{b_i\}_{i \geqslant 1}$ is a weakly decreasing sequence of non-negative integers. In [9] Banerjee, Beyarslan and Hà conjectured that for any edge ideal, $\{b_i\}_{i \geqslant 1}$ is a weakly decreasing sequence (see [9, Conjecture 7.11]). For the edge ideal of any dumbbell graph with $l \leqslant 2$, we prove $b_i = b_1$ for all $i \geqslant 1$. However, we expect $b_i \leqslant b_1$ for all $i \geqslant 1$ for any graph.

Remark 7.69. *From Theorem 7.27 and Theorem 7.39, for any* $l \leqslant 2$ *we have that*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) \geqslant \left\lfloor \frac{n+m+l+1}{3} \right\rfloor.$$

The previous inequality is not satisfied when $l \geqslant 3$, because $\operatorname{reg} I(C_4 \cdot P_3 \cdot C_4) = 3$ and $\lfloor \frac{4+4+3+1}{3} \rfloor = 4$.

As recalled earlier, we use the notation of even-connection from Banerjee [8, Theorem 5.2]. The following lemma is important in our treatment of the even-connected vertices, and its proof is similar to [8, Lemma 6.13].

Lemma 7.70. *Let* G *be a graph. As in Remark 7.10, let* G' *be the graph associated to* $(I(G)^{q+1} \colon e_1 \cdots e_q)^{pol}$. *Suppose* $u = p_0, p_1, \ldots, p_{2s+1} = v$ *is a path that even-connects* u

and v *with respect to the* q*-fold* $e_1 \cdots e_q$*. Then, we have*

$$\bigcup_{i=0}^{2s+1} N_{G'}[p_i] \subset N_{G'}[u] \cup N_{G'}[v].$$

Proof. Let U be the set of vertices $U = \{p_0, p_1, \ldots, p_{2s+1}\}$. For each $1 \leqslant k \leqslant s$ we have that $p_{2k-1}p_{2k} = e_{j_k}$ for some $1 \leqslant j_k \leqslant q$, i.e. u and v are even connected with respect to the s-fold $e_{j_1}e_{j_2}\cdots e_{j_s}$.

Let w be a vertex even-connected to some vertex $z \in U$ with respect to the q-fold $e_1 \cdots e_q$. Then, there exists a path $z = r_0, r_1, \ldots, r_{2t+1} = w$ that even-connects z and w with respect to the q-fold $e_1 \cdots e_q$. Let i be the largest integer such that $r_i \in U$. From the fact that $r_0 = z \in U$, we have that the integer i is well defined and $i \geqslant 0$. Let k be an integer such that $p_k = r_i$.

The proof is now divided into four different cases depending on $i \bmod 2$ and $k \bmod 2$. When i and k are both odd integers, we have that $r_i r_{i+1}$ is equal to some edge of $\{e_1, e_2, \ldots, e_q\}$ and that $p_{k-1}p_k$ is not equal to any edge of $\{e_{j_1}, e_{j_2}, \ldots e_{j_s}\}$. By the definition of i, we have

$$\{r_{i+1}, r_{i+2}, \ldots, r_{2t+1}\} \cap U = \emptyset.$$

So, in this case, it follows that

$$u = p_0, \ldots, p_{k-1}, p_k = r_i, r_{i+1}, \ldots, r_{2t+1} = w$$

is a path that even-connects u and w with respect to the q-fold $e_1 \cdots e_q$.

The other three cases follow in a similar way. Therefore, we have that if w is even-connected to some $z \in U$, then w is even-connected to either u or v.

Now, we only need to prove that any $w \in N_G[z]$ for some $z \in U$ is even-connected to either u or v. This part is simple, if $z = p_{2j}$ then $u = p_0, \ldots, p_{2j} = z, w$ is a path that even-connects u and w. Otherwise, if $z = p_{2j-1}$ then $w, z = p_{2j-1}, \ldots, p_{2s+1} = v$ is a path that even-connects w and v. So, the proof follows. $\square$

The next lemma is similar to [10, Lemma 5.1], but adapted to the current setting of a dumbbell.

Lemma 7.71. *Let* $G = C_n \cdot P_l \cdot C_m$*. If* $(I(G)^{q+1} \colon e_1 \cdots e_q)$ *is not a square-free monomial ideal and* G' *is the associated graph, then there exists a vertex z which is even-connected to itself. Then,* G' *has a leaf and* $N_{G'}[z]$ *contains one of the two cycles. In particular, if we denote the corresponding leaf by* e*, then* G'_e *is an induced subgraph of a unicyclic graph.*

Proof. Suppose $z = p_0, p_1, \ldots, p_{2l+1} = z$ is an even-connection of z with itself. Let

$$0 \leqslant a < b \leqslant 2l + 1$$

be integers such that $p_a, p_{a+1}, \dots, p_b = p_a$ is an even-connection and $b - a$ is minimal. Then, $p_a, p_{a+1}, \dots, p_b = p_a$ is a simple closed path lying on $C_n \cdot P_l \cdot C_m$ and so it is necessarily equal to either C_n or C_m.

Finally, Lemma 7.70 implies that $N_{G'}[z]$ contains either C_n or C_m. $\square$

Lemma 7.72. *Let* $G = C_n \cdot P_l \cdot C_m$ *with* $l \leqslant 2$ *and* H *be a graph such that* G *is a subgraph of* H *with the same set of vertices (i.e.,* $V(H) = V(G)$ *and* $E(H) \supseteq E(G)$*). For any two vertices* $u, v \in H$ *such that* $\{u, v\} \notin E(G)$*, we have that*

$$\operatorname{regI}(H \setminus (N_H[u] \cup N_H[v])) \leqslant \operatorname{regI}(G) - 1.$$

Proof. Let $K = N_G[u] \cap N_G[v]$. We divide the proof according to the cardinality $|K|$ of K. Notice that for the dumbbell G we always have $0 \leqslant |K| \leqslant 2$.

Since $H \setminus (N_H[u] \cup N_H[v])$ is an induced subgraph of $H \setminus (N_G[u] \cup N_G[v])$, from Theorem 7.7$(i)$, it is enough to prove that $\operatorname{regI}(H \setminus (N_G[u] \cup N_G[v])) \leqslant \operatorname{regI}(G) - 1$.

Step 1. Suppose that $|K| = 0$. Then, the graph $H \setminus (N_G[u] \cup N_G[v])$ is obtained by deleting at least 6 vertices, and so $|H \setminus (N_G[u] \cup N_G[v])| \leqslant |G| - 6 \leqslant n + m + l - 8$. Note that we can add two vertices to $H \setminus (N_G[u] \cup N_G[v])$ and connect them in such a way that we obtain a Hamiltonian path. Let L be a graph obtained by adding two vertices and certain edges connecting these two new vertices, in such a way that L has a Hamiltonian path. Since $|L| \leqslant n + m + l - 6$, Theorem 7.22 yields

$$\operatorname{regI}(L) \leqslant \left\lfloor \frac{n+m+l-5}{3} \right\rfloor + 1 = \left\lfloor \frac{n+m+l+1}{3} \right\rfloor - 1,$$

and by applying Remark 7.69, we get $\operatorname{regI}(L) \leqslant \operatorname{regI}(G) - 1$. Since $H \setminus (N_G[u] \cup N_G[v])$ is an induced subgraph of L, Theorem 7.7(i) implies that $\operatorname{regI}(H \setminus (N_G[u] \cup N_G[v])) \leqslant \operatorname{regI}(G) - 1$.

Step 2. Suppose that $|K| = 1$. Here the proof follows along the same lines of Step 1. In this case the graph $H \setminus (N_G[u] \cup N_G[v])$ is obtained by deleting at least 5 vertices. Now, note that we can add one vertex to $H \setminus (N_G[u] \cup N_G[v])$ and connect it in such a way that we obtain a Hamiltonian path. Let L be a graph obtained by adding one vertex and certain edges connecting this new vertex, such that L has a Hamiltonian path. Since $|L| \leqslant (n + m + l - 2) - 5 + 1 = n + m + l - 6$, then the rest of the proof follows as in Step 1.

Step 3. Suppose that $|K| = 2$. In this case, note that one of the cycles is necessarily equal to C_4, say $C_n = C_4$, and that $u, v \in C_4$ with $\{u, v\} \notin E(G)$. Hence, it follows that $H \setminus (N_G[u] \cup N_G[v])$ has a Hamiltonian path with $\leqslant m$ vertices if $l = 2$ and $\leqslant m-1$ vertices if $l = 1$. From Theorem 7.22 and Remark 7.69, then we have $\operatorname{regI}(H \setminus (N_G[u] \cup N_G[v])) \leqslant \operatorname{regI}(G) - 1$.

So, the proof follows. $\square$

Theorem 7.73. *Let* $G = C_n \cdot P_l \cdot C_m$ *with* $l \leqslant 2$ *and* $I = I(G)$ *be its edge ideal, then*

$$\operatorname{reg}(I^{q+1} : e_1 \cdots e_q) \leqslant \operatorname{regI}$$

for any $1 \leqslant q$ *and any edges* $e_1, \dots, e_q \in E(G)$.

Proof. We split the proof into two cases.

Case 1. First, suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is a square-free monomial ideal. In this case $(I^{q+1} \colon e_1 \cdots e_q) = I(G')$ where G' is a graph with $V(G) = V(G')$ and $E(G) \subseteq E(G')$. Let $E(G') = E(G) \cup \{a_1, \ldots, a_r\}$, then each edge a_i is induced from even-connecting two different vertices (i.e., each a_i is not a whisker). By Theorem 7.7, we have

$$\operatorname{reg} I(G') \leqslant \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G'_{a_1}) + 1\}$$

Since $a_1 \notin E(G)$, Lemma 7.72 implies that $\operatorname{reg} I(G'_{a_1}) + 1 \leqslant \operatorname{reg} I(G)$.

In the same way, for any subgraph $H = G' \setminus \{a_1, \ldots, a_i\}$, since $V(H) = V(G)$ and $E(H) \supseteq E(G)$, Lemma 7.72 also gives us that

$$\operatorname{reg}(I(H_{a_{i+1}})) + 1 \leqslant \operatorname{reg}(I(G)).$$

By continuing this process, we get $\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G)$.

Case 2. Suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is not square-free and G' is the graph associated to $(I^{q+1} \colon e_1 \cdots e_q)^{\mathrm{pol}}$. Let $\{b_1, b_2, \ldots, b_s\}$ be the subset of edges of $E(G') \setminus E(G)$ that are generated by square monomials (i.e., each b_i is a whisker).

From Theorem 7.7 we have the inequality

$$\operatorname{reg} I(G') \leqslant \max\{\operatorname{reg} I(G' \setminus b_1), 1 + \operatorname{reg} I(G'_{b_1})\}.$$

Lemma 7.71 implies that one of the cycles is deleted from G'_{b_1}, then there exists an edge $e \in G$ such that $d(e, G'_{b_1}) \geqslant 2$. So, for such an edge e we get that the disjoint union $G'_{b_1} \cup e$ is an induced subgraph of $G' \setminus b_1$. Thus, Theorem 7.7 and Corollary 7.14 yield that

$$\operatorname{reg}(I(G'_{b_1})) + 1 = \operatorname{reg}(I(G'_{b_1} \cup e)) \leqslant \operatorname{reg}(I(G' \setminus b_1)).$$

Therefore, we obtain that $\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G' \setminus b_1)$.

By applying the same argument, it follows that

$$\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G' \setminus b_1) \leqslant \operatorname{reg} I(G' \setminus \{b_1, b_2\}) \leqslant \cdots \leqslant \operatorname{reg} I(G' \setminus \{b_1, \ldots, b_s\}).$$

Since the graph $G' \setminus \{b_1, \ldots, b_s\}$ has no whiskers, then Step 1 implies that

$$\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G' \setminus \{b_1, \ldots, b_s\}) \leqslant \operatorname{reg} I(G).$$

So, the proof follows. $\square$

Remark 7.74. *The previous theorem is a generalization of a work done by Gu in* [61] *for the case* $l = 1$.

Theorem 7.75. *For the dumbbell graph* $C_n \cdot P_l \cdot C_m$ *with* $l \leqslant 2$ *and any* $q \geqslant 1$*, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q \geqslant 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2.$$

Proof. Using the inequality $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geqslant 2q + \nu(C_n \cdot P_2 \cdot C_m) - 1$ of Theorem 7.21, for the cases where $\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1$ we get the expected inequality. We divide the proof in two halves, the cases $l = 1$ and $l = 2$.

Case 1. Let $l = 1$. We only need to focus on the case where $n, m \equiv 2 \pmod 3$. Let H be the induced subgraph of $C_n \cdot P_1 \cdot C_m$ mentioned in the proof of Theorem 7.30, i.e.

$$H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\} = P_{n-1} \cdot C_m.$$

Using Theorem 7.27, Proposition 7.26 and the modularity $n, m \equiv 2 \pmod 3$, we can check that

$$\nu(H) = \nu(C_n \cdot P_1 \cdot C_m)$$

and that

$$\nu(H) = \nu(H \setminus \Gamma_H(C_m)).$$

From Theorem 7.30 and [4, Theorem 1.2] we get

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) = \nu(C_n \cdot P_1 \cdot C_m) + 2 = \nu(H) + 2 = \operatorname{reg} I(H).$$

Since H is an induced subgraph of $C_n \cdot P_1 \cdot C_m$, then from [4, Theorem 1.1] and [10, Corollay 4.3] we get the inequality

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m)^q \geqslant \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = 2q + \operatorname{reg} I(C_n \cdot P_1 \cdot C_m) - 2.$$

Case 2. Let $l = 2$. We only need to focus on the cases where $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. We take the same induced subgraph H as in Lemma 7.35. The induced subgraph $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ of $C_n \cdot P_2 \cdot C_m$ is given as the union of a path of length $n - 1$ and the cycle C_m, i.e., $H = P_{n-1} \cup C_m$.

By Theorem 7.36, for the cases $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$, we have

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \nu(C_n \cdot P_2 \cdot C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2,$$

and from Corollary 7.14, Theorem 7.15 and Theorem 7.20 we obtain

$$\operatorname{reg} I(H) = \operatorname{reg}(I(P_{n-1})) + \operatorname{reg}(I(C_m)) - 1 = \nu(P_{n-1}) + \nu(C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Hence, we get $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \operatorname{reg} I(H)$. Finally, using [4, Theorem 1.1] and [10, Corollary

4.3], we get the inequality

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geqslant \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = 2q + \operatorname{reg} I(C_n \cdot P_2 \cdot C_m) - 2.$$

So, the proof follows. $\square$

Theorem 7.76. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leqslant 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geqslant 1$.

Proof. It follows by Theorem 7.73, Theorem 7.11 and Theorem 7.75. $\square$

Remark 7.77. *One may ask whether*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

always holds for given n, m, l and q. Unfortunately, it is no longer true for any n, m, l and q as it can be seen from the following example:

$$6 = \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5)^2 < 4 + \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5) - 2 = 7.$$

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